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ANTIFERROMAGNETIC PROPERTIES OF COBALTOUS OXIDE

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The magnetic susceptibility of CoO was investigated between 100 and 700° K. A sharp maximum was found at 292° K. The dependence of susceptibility on field in the antiferromagnetic state and in the region of the antiferromagnetic Curie point was determined.

1. One of the characteristic properties of antiferromagnets is the slow increase in magnetic susceptibility with increasing temperature, below the antiferromagnetic Curie point (Θ_{af}), and its decrease above this temperature, according to the Curie-Weiss law. Landau[1] was the first to explain theoretically this variation of magnetic susceptibility with temperature for antiferromagnets.

In the present work we have studied the temperature dependence of the magnetic susceptibility of polycrystalline specimens of cobaltous oxide. In other studies[2-4] of this substance it was shown that CoO has antiferromagnetic properties, but a sharp maximum in the susceptibility-temperature curve was only found in La Blanchetais' work.[4] Singer[5] studied the magnetic properties of a CoO single crystal and also found a susceptibility maximum. In all the work mentioned, the dependence of magnetic susceptibility χ on the strength of the magnetic field was hardly examined. The existing data on the dependence of χ on H for other antiferromagnets are inconsistent[4,6,7] and these facts have not yet been completely explained. It therefore seemed of great interest to study the magnetic susceptibility of cobaltous oxide over a wide range of magnetic field and for different temperatures.

2. The measurements of susceptibility were made with an apparatus based on the Faraday-Sucksmith method.[8] In this method, the force acting on a specimen, placed in a strong inhomogeneous magnetic field, is measured by the deformation of an elastic ring made of beryllium bronze. For a specimen mass of 10 — 25 mg the sensitivity of the apparatus was a susceptibility of ~ 10^{-7} per scale division.

Cobaltous oxide was investigated over the temperature range 100 — 700° K. The low temperatures were obtained by using nitrogen vapor which flowed around the specimen under examination. A small nichrome heater was placed in a dewar containing liquid nitrogen, and its temperature determined the rate at which the nitrogen evaporated. The nitrogen vapor passed from the dewar vessel into a double-walled glass tube in which the specimen was hung freely on a quartz thread. A copper tube of smaller size was placed inside the glass tube so that there should be a uniform temperature over a certain length. Temperatures above room temperature were obtained by using a small oven wound of bifilar platinum wire on a porcelain tube of diameter 6 mm and length ~ 70 mm. The temperature was measured with a copper-constantan thermocouple, previously calibrated.

The results of La Blanchetais[4] show that the magnetic properties of cobaltous oxide depend markedly on the method of preparing the specimens. For this reason great attention was paid

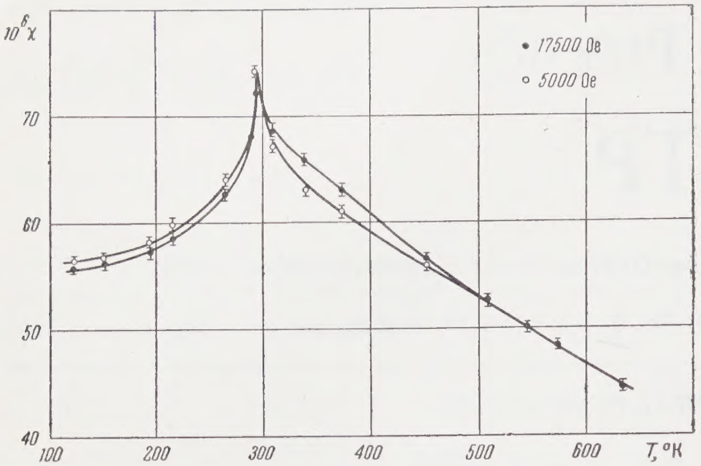

FIG. 1. The temperature dependence of magnetic susceptibility.

to specimen preparation. Cobalt nitrate was used, heated in air to 1000° C for three hours. Specimens were pressed from the powder so obtained under a pressure of 2000 kg/cm^2, and were heated in vacuum (10^{-3} mm Hg) for three hours at 820° C. We should remark that specimens prepared in this way were practically identical in magnetic properties.

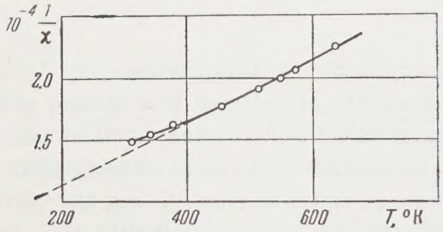

FIG. 2. The temperature dependence of the inverse magnetic susceptibility, H = 17,500 oe.

3. As stated above, the magnetic susceptibility was measured over the wide temperature range from 100 to 700° K. The temperature dependence of the specific susceptibility remains practically constant at low temperatures. With increasing temperature it increases steadily and has a sharp maximum at 292° K. With further increase of temperature χ rapidly decreases.

To determine the dependence of χ on H, the susceptibility was measured for various values of the magnetic field up to ~ 18,000 oe. Below the antiferromagnetic Curie point, i.e., in the antiferromagnetic state, the susceptibility decreases with increasing field, while at temperatures above Θ_{af}, near the transition point, χ on the contrary increases.

La Blanchetais[4] found the same dependence of χ on H for cobaltous oxide, but in her work the range of fields was small, so that the behavior was less marked. It is interesting that the difference between the susceptibilities for different fields decreases on approaching the Curie point and increases with decreasing temperature; according to our data this difference is greatest at T = 100° K.

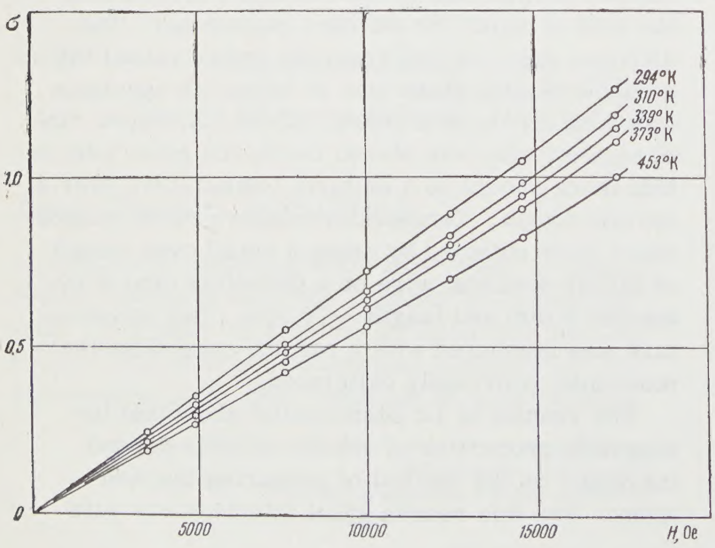

FIG. 3. Magnetization curves at temperatures above Θ_{af}.

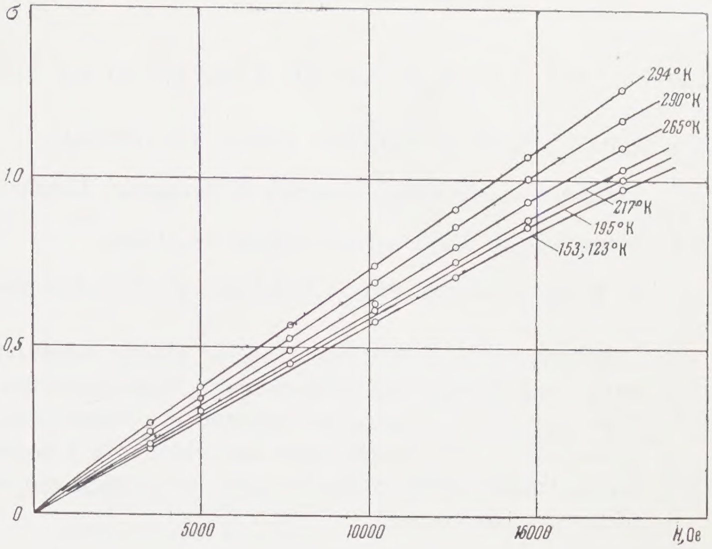

FIG. 4. Magnetization curves at temperatures below Θ_{af}.

We should point out that the temperature of the antiferromagnetic Curie point, $\Theta_{af} = 292°K$ is independent of the strength of the external field. The value of Θ_{af} obtained agrees well with the results of measurements of other physical quantities.[9]

The susceptibility at absolute zero can be obtained by extrapolation of the curve of the temperature dependence of susceptibility shown in Fig. 1 to $T = 0°K$. This is $\sim 55 \times 10^{-6}$ from our measurements. According to the theory of Van Vleck,[10] the susceptibility at $T = 0°K$ of an antiferromagnet with two sublattices is $2/3$ of the susceptibility at $T = \Theta_{af}$. For the case of several sublattices the value of χ_0/χ_{af} should be somewhat higher according to Anderson;[11] in fact we found $\chi_0/\chi_{af} = 0.76$.

Figure 2 shows the temperature dependence of $1/\chi$ above the antiferromagnetic Curie point. It can be seen that this dependence follows the Curie-Weiss law above 470°K. The values of the Curie-Weiss constant C, the atomic magnetic moment P_p and the paramagnetic Curie point Θ_p determined from our experimental data were $C = 3.05$, $P_p = 4.97$, $\Theta_p = -270°C$. The values of magnetic moment and of the paramagnetic Curie point agree well with the data of La Blanchetais[4] who obtained $P_p = 4.96$, $\Theta_p = -280°C$.

There is a temperature region above the antiferromagnetic Curie point $(T \gtrsim \Theta_{af})$ up to $\sim 470°K$ where the magnetic susceptibility is a function both of the temperature and of the magnetic field strength, as was also found by La Blanchetais. The isothermal magnetization curves in the antiferromagnetic state and in the region of the antiferromagnetic Curie point are given in Figs. 3 and 4. Near Θ_{af} the dependence of specific magnetization on H is practically linear. On lowering the temperature in the antiferromagnetic state, the isotherms start to curve gradually. The curves are then concave to the H axis. In the paramagnetic region $(T \gtrsim \Theta_{af})$ there is an analogous small curvature, but in the opposite direction.

4. The study of the temperature dependence of the magnetic susceptibility of cobaltous oxide showed that this substance takes on well defined antiferromagnetic properties, which appear at $T = 292°K$ in the form of a sharp maximum in the $\chi(T)$ dependence. In the paramagnetic region the susceptibility of CoO follows the Curie-Weiss law, which is observed in most antiferromagnets. A small dependence of magnetic susceptibility on field strength was found both above and below the temperature of the antiferromagnetic transition.

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ANGULAR DISTRIBUTION OF ELASTICALLY SCATTERED 14.5-Mev NEUTRONS

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The differential cross sections for elastic scattering of 14.5-Mev neutrons on silver, mercury, and bismuth were measured. Neutrons from the T(d, n)α reaction were scattered from spherical targets and counted in coincidence with the accompanying α particles by means of a pulse height-time analyzer with a resolving time of 5×10^{-9} sec. The experimental differential cross sections are compared with cross sections calculated on the basis of the optical model.

1. INTRODUCTION

In the last few years, a large number of measurements of differential cross sections for elastic scattering of protons and neutrons by nuclei have been published. These measurements have been performed over a wide energy range for the scattered nucleons and for a substantial number of nuclei. Aside from its purely practical purposes (neutrons in the fission energy range), this subject has become interesting because of the success of the optical model of the nucleus,[1] particularly its later modifications,[2,3] which agree quite well with experiment. For comparison with the optical model, neutron experiments are preferable because there is no Coulomb interaction with the nucleus. The 14.5-Mev neutrons from the T(d, n)He^4 reaction have often been used for this purpose,[4-9] because, on the one hand, the reaction is easy to realize experimentally and the neutrons are monoenergetic and are accompanied by α particles (which are utilized in a considerable number of the experiments), and, on the other hand, theoretical considerations lead one to expect that at this neutron energy processes involving compound nucleus formation should make a very small contribution to the total scattering. However, the data on angular distributions from elastic scattering of 14.5-Mev neutrons are as yet available for a limited number of nuclei.

2. APPARATUS

The technique known as electronic collimation of the neutrons was applied for the measurements. This collimation technique is based on the correlation of the neutron with its accompanying α particle and is accomplished by means of a pulse height-time analyzer.[10]

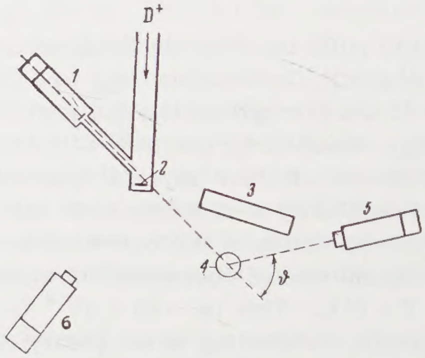

FIG. 1. Experimental arrangement: 1 – α particle counter, 2 – tritium target, 3 – shield, 4 – scatterer, 5 – detector, 6 – monitor.

The experimental arrangement is shown in Fig. 1. The neutrons were produced from the T(d, n)α reaction in a neutron generator.[11] A scintillation counter with a stilbene crystal (diameter 3.5 cm, length 2.4 cm) and an FÉU-33 photomultiplier was used as the neutron detector. The output of the photomultiplier, on which an amplitude resolution of 7% was attained with good output pulse shape, was sent through a fast electronic gate into a pulse-height analyzer.

The accompanying α particles were recorded by a thin stilbene crystal (0.04 cm) on the photocathode of an FÉU-33. An aluminum foil (0.9 mg/cm^2) was between the crystal and the target.

The neutrons detected in coincidence with particles are in a solid angle determined by the angle subtended by the α particle counter. The "collimation curve," Fig. 2, obtained by measuring the neutron intensity as a function of angle around the tritium target, gives 9° for the half-width of the collimated neutron beam. In addition, a copper shield was placed between the target and the neutron detector.

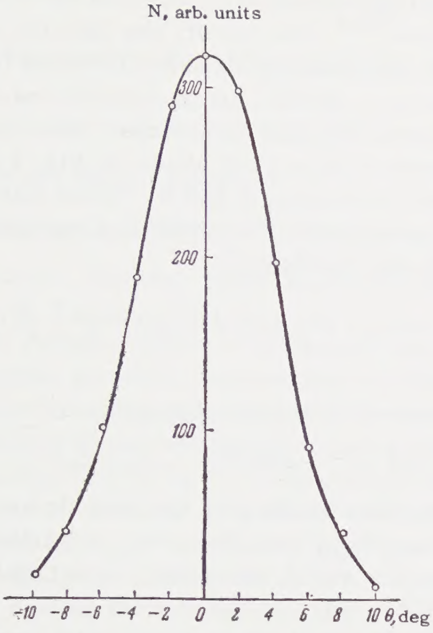

FIG. 2. Collimation curve (N is the number of counts).

The scatterers were spheres, 4 cm in diameter (the mercury was in a thin-walled container).

A block diagram of the pulse height-time analyzer is shown in Fig. 3. The pulses from the anode of the photomultiplier are limited by a 6Zh5P pentode, clipped by a short-circuited coaxial cable, and then go to the coincidence diode. A delay line ensures that the neutron pulses coincide with the pulses from the accompanying α particles.

The pulses for the pulse height analyzer are taken from the tenth dynode of the photomultiplier and after going through an amplifier and a discriminator are fed into a triple coincidence circuit with a resolution of 5×10^{-7} sec. The slow coincidence circuit is useful for eliminating counts due to inelastic neutrons, γ rays, and photomultiplier noise.

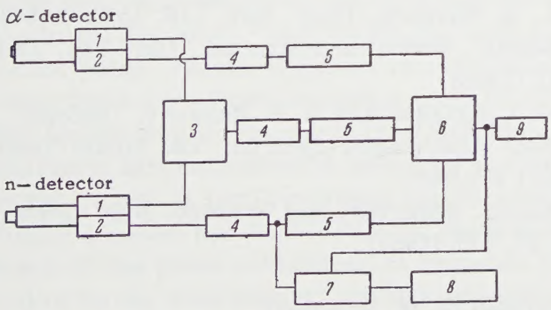

FIG. 3. Block diagram of the pulse height-time analyzer. 1 – limiting pentode, 2 – cathode follower, 3 – fast coincidence circuit, 4 – amplifier, 5 – discriminator, 6 – triple coincidence circuit, 7 – amplitude-preserving gate, 8 – pulse height analyzer, 9 – counter.

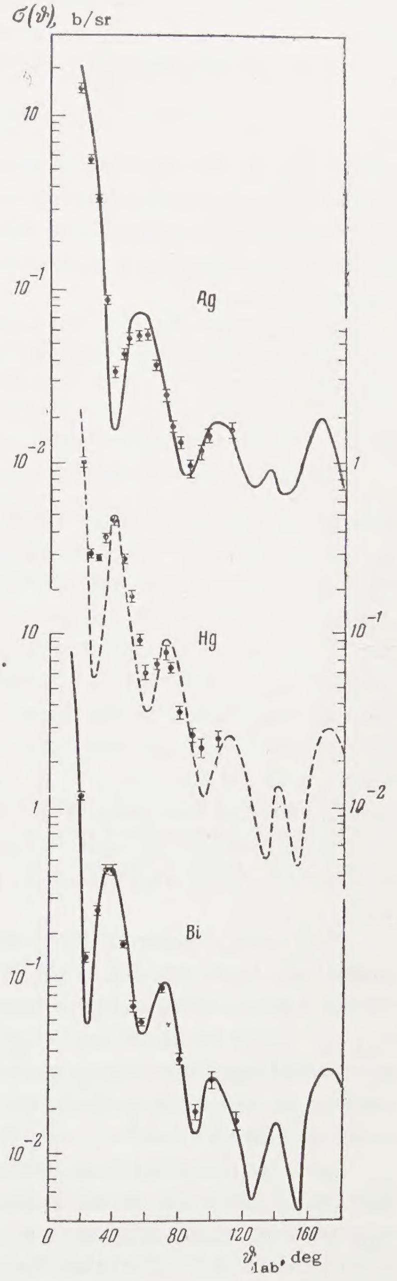

FIG. 4. Elastic scattering differential cross sections for 14.5-Mev neutrons on silver, mercury, and bismuth. The curves are the results of optical-model calculations.

The resolving time of the analyzer can be controlled by a retarding voltage on the coincidence diode and also by changing the setting of the discriminator in the time channel. In the present experiment it was a 4×10^{-9} sec. By increasing the discriminator setting, a resolving time of 1×10^{-9}sec can be attained, but the efficiency of the circuit is then strongly decreased.

3. MEASUREMENTS AND COMPUTATIONS

Measurements were carried out on silver, mercury, and bismuth for scattering angles, ϑ, from 20° to 110°.

In order to compute the absolute differential cross section, it is necessary to determine experimentally the ratio

$$S(\vartheta) = (N_s - N_b)/N_d, \qquad (1)$$

where N_s is the number of counts due to neutrons elastically scattered through the angle ϑ, N_b is the number of counts due to noise, and N_d is the number of counts in a measurement in the direct neutron beam.

N_s, N_b, and N_d are normalized by the monitor. The differential cross section is calculated from the formula

$$\sigma_{el}(\vartheta) = S(\vartheta)\,[R_1R_2/(R_1+R_2)]^2 \times \exp\{n\sigma_{in}d\}[NB(E_n)\eta]^{-1}, \qquad (2)$$

Where R_1 is the distance from source to scatterer, R_2 is the distance from scatterer to detector, n is the number of nuclei per cubic centimeter in the scatterer, σ_{in} is the inelastic cross section, d is the thickness of the scatterer, N is the number of scattering nuclei, $B(E_n)$ takes into account the energy sensitivity of the detector, η takes into account the collimation configuration of the neutron beam, and $[R_1R_2/(R_1+R_2)]^2$ is the coefficient which reduces the ratio $S(\vartheta)$ to unit solid angle.

The threshold for discriminating against inelastically scattered neutrons was taken to be 11 Mev.

The configuration of the collimated neutron beam was taken into account in computing the cross section; this allowed the absolute value of $\sigma_{el}(\vartheta)$ to be obtained and eliminated the necessity of normalizing the experimental results to the theoretical curve, as was done, for example, in the work of Berko et al.[5]

The experimental data were corrected for depletion of the beam in the scatterer; the use of σ_{in} instead of the total cross section σ_t in these corrections partially takes into account multiple scattering. Corrections for finite geometry were made by a method similar to that used in the work of Meier et al.[12]

4. RESULTS

The experimental results are shown in Fig. 4 along with theoretical curves. The angles are given in the laboratory coordinate system. The statistical errors, varying from 4% at scattering angles less than 50° to 7 or 8% at larger angles, are shown in the figure. The results in bismuth are in good agreement with those of several other experiments.[5,7,9] For silver, the data for scattering angles less than 90° are the first that have been obtained. Nauta's measurements on mercury are less accurate than the present ones.

The theoretical curves shown in Fig. 4 are taken from references 5 and 6. These curves were calculated with the potential proposed by Bjorklund and Fernbach:[2]

$$V = V_{CR}p(r) + iV_{CJ}q(r) + V_{SR}(\hbar/mc)^2\frac{1}{r}\frac{dp}{dr}(\sigma\mathbf{l}),$$

where

$$p(r) = [1 + \exp\{(r - R_0)/a\}]^{-1},$$

$$q(r) = \exp[-(r - R_0)^2/b^2], \qquad R_0 = r_0A^{1/3}. \qquad (3)$$

The parameters which give the best fit for 14.5 Mev neutrons are $V_{CR} = 44$ Mev, $V_{CJ} = 11$ Mev, $V_{SR} = 8.3$ Mev, a = 0.65 f, b = 0.98 f, $r_0 = 1.25$ f.

Since there were no theoretical curves for mercury at our disposal, the experimental points in this case are compared with the curve for lead. Such a comparison is reasonable, since mercury and lead do not differ much in atomic weight.

The agreement of the calculated curves and the experimental points is sufficiently good.

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MAGNETO-OPTICAL PROPERTIES OF GARNET FERRITES IN THE INFRARED REGION

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The Faraday effect at infrared frequencies was measured in yttrium and holmium garnet ferrites at room temperature and liquid nitrogen temperature. The sign of the Faraday effect in holmium garnet is reversed on passing through the compensation point, due to the rotation of the ferrimagnetic spin system in a magnetic field. An absorption band at $\lambda = 1.96\mu$ due to $^5I_8 \rightarrow {}^5I_7$ electronic transitions between ground-state sublevels of holmium ions was detected by both optical and magneto-optical means. By comparing the experimental data for yttrium and holmium garnets it was possible to detect the rotation of the plane of light polarization due to one of the magnetic sublattices of the ferrimagnetic substance.

GARNET ferrites, which are transparent in the infrared region, contain three magnetic sublattices (d, a, and c) consisting of 24 Fe^{3+} ions in d tetrahedral sites, 16 Fe^{3+} ions in a octahedral sites, and 24 rare-earth or yttrium ions in c sites.[1] The magnetic moment of the c sublattice is antiparallel to the total magnetic moment of the iron ions. Most garnet ferrites exhibit a so-called compensation point, the temperature at which the spontaneous magnetization of the garnet vanishes due to equality of the magnetic moments of sublattices c and d + a. At higher temperatures the magnetization of the iron sublattices is dominant, while at lower temperatures the magnetization of the rare earth sublattice predominates. A compensation point does not exist when the 24 c sites are occupied by nonmagnetic ions or by rare-earth ions with a small magnetic moment.

We investigated two garnet ferrites — $Y_3Fe_5O_{12}$ without a compensation point, the Faraday effect in which has been measured over a broad infrared region in earlier work,[2] and $Ho_3Fe_5O_{12}$ with a compensation point. Holmium garnet is very suitable for study because its compensation point is 136° K, so that approximately the same considerable amount of saturation magnetization is exhibited at room temperature and at liquid nitrogen temperature.

Figure 1 shows the distribution of the magnetic moments in the three sublattices of holmium garnet according to the Néel theory, and the orientations of these moments in an external magnetic field both above and below the compensation point. More detailed information concerning the crystallographic and magnetic structures of garnet ferrites can be found in Belov's monograph.[3] Since the yttrium ion is nonmagnetic, the magnetic moments of the d and a sublattices in Fig. 1 determine the absolute values of saturation magnetization in yttrium garnet. In other words, when we investigate yttrium garnet we are studying two magnetic sublattices that are contained in all garnets. In the other garnets the additional sublattice consists of magnetically active rare-earth ions.

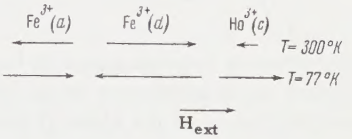

FIG. 1. Orientation of the magnetic moments of the three sublattices of holmium garnet ferrite in an external magnetic field. The lengths of the vectors are proportional to the magnetic moments of the sublattices.

Electrons in the well-screened f shell of rare-earth ions interact only weakly with the electrons of neighboring atoms. This circumstance facilitates the study of interactions in garnet ferrite crystals, since the energy level scheme of an isolated rare-earth ion can be used in zero approximation. Moreover, the weakness of the given reaction results in narrow absorption lines associated with transitions between f levels. The detection and study of narrow lines in solids, especially in ferromagnets, is important for quantum radiophysics.

The garnet samples were polished single-crystal plates about 100μ thick with about 0.1 cm^2 lateral face area. The monochromator was an IKS-11 spectrograph. Infrared light polarized by a selenium mirror passed through the single-crystal sample that had been magnetized perpen-

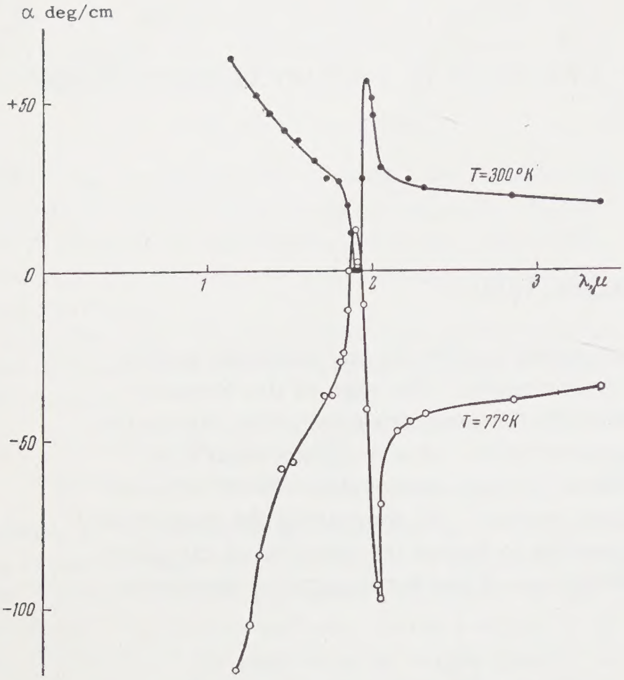

FIG. 2. Faraday effect in holmium garnet at two temperatures, above and below the compensation point. Ordinates represent the specific rotation α_Φ of the plane of polarization in degrees per cm.

dicular to its surfaces by a small electromagnet generating a field H = 1500 oe. The analyzer consisted of a few silver chloride plates[4] and was oriented at 45° to the polarizer. This angle yields maximum variation of transmitted light intensity as the plane of polarization is rotated.

For measurements at liquid nitrogen temperature the sample was placed in a cryostat with glass windows for light transmission. The light sources were a Nernst glower and a tungsten incandescent lamp; the detectors were a vacuum thermocouple and FÉU-22 photomultiplier. Absorption was determined by comparing measurements with and without the garnet. No correction was made for reflection because of lack of data regarding the refractive index of garnets.

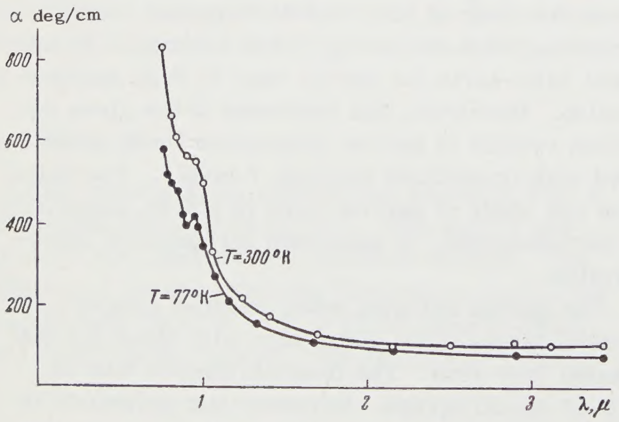

FIG. 3. Faraday effect in yttrium garnet at room temperature and at liquid nitrogen temperature.

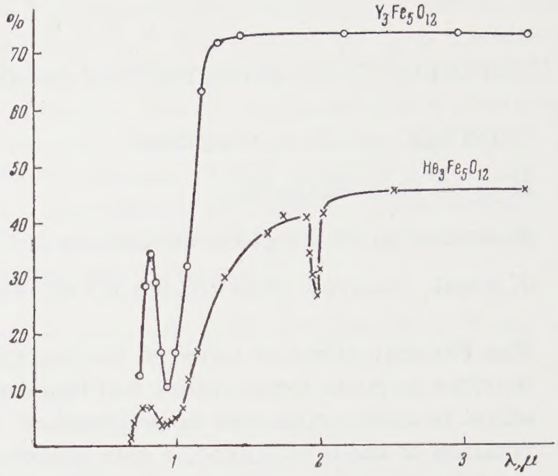

FIG. 4. Infrared transmission coefficients (%) of yttrium and holmium garnets.

Figure 2 shows measurements of the Faraday effect in holmium garnet at two temperatures, above and below the compensation point. The curves are distinguished by the different directions in which the plane of light polarization is rotated by the garnet at room temperature and at 77°K. The reversal of the Faraday effect results from the reorientation of the atomic magnetic moments of all garnet sublattices on passing through the compensation point. Below this point the magnetic moment of the holmium sublattice, which becomes larger than that of the iron sublattices, is oriented parallel to the external magnetic field, and rotates the ferrimagnetic spin system through 180° (Fig. 1).

The observed effect is direct evidence for the correctness of the magnetic structure scheme of garnet ferrites proposed by Néel.[1] In addition, the sign reversal of the Faraday effect indicates that the rotation of the plane of light polarization is not associated with the resultant spontaneous magnetization of a garnet ferrite as a whole, which always assumes the direction of the external magnetic field, but rather with the magnetic moments of the separate ferrite sublattices. For example, measurements of the rotation of the polarization plane of microwaves revealed no reversal of rotation on passing through the compensation point, since this effect, like the great majority of physical effects in ferrimagnetics, depends on the resultant magnetic moment.

Figure 3 shows the Faraday effect in yttrium garnet at two given temperatures. The rotation of the polarization plane is not reversed because the 24 c sites are occupied by nonmagnetic ions.

Clogston[5] has attempted to formulate a theory of the Faraday effect in yttrium garnet. The characteristic frequencies of the dispersion curves of the Faraday effect are determined by forbidden transitions in Fe^{3+} ions. The oscillator strengths of these transitions and the magnitude of the Faraday effect depend on the existence of higher-lying levels to which electric dipole transitions are allowed. Therefore a quantitative check of the theory requires knowledge of the energy spectrum of iron ions in the garnet crystal.

The curves in Fig. 2 show a pronounced anomaly in the Faraday effect at $\lambda = 2\mu$. The measurement of absorption by holmium garnet showed that this anomaly is associated with the absorption band at $\lambda = 1.96\mu$ (Fig. 4). This absorption band is very probably attributable to an electronic transition ($^5I_8 \rightarrow {}^5I_7$) between the first two sublevels of the lowest multiplet level, which had been observed previously in Ho^{3+} ions at $\lambda = 1.94\mu$.[6] Yttrium garnet exhibits an absorption band at 0.92μ (Fig. 4), associated with an electronic transition in Fe^{3+}. Absorption by holmium garnet at wavelengths from 0.9 to 1.2μ is due to the indicated transition in Fe^{3+} and to the $^5I_8 \rightarrow {}^5I_6$ transition in Ho^{3+} ($\lambda_0 = 1.14\mu$).[6] Thus in the characteristic frequency region optical methods enable us to determine the fraction of light absorption and of the rotation of the polarization plane that is due to the ions of one of the garnet magnetic sublattices. This opens wide possibilities in the study of ferrimagnetics.

The 0.5-mm slits used for our measurements correspond to a wavelength interval of $\sim 0.05\mu$ at $\lambda = 2\mu$. It will be necessary in the future to improve the monochromaticity of light used in measuring the Faraday effect. Higher resolving power will enable us to improve the shape of the Faraday effect dispersion curve, and to study the fine structure and polarization of the separate components.

Figure 2 shows the measurements of the Faraday effect in holmium garnet only up to 3.5μ. Measurements at room temperature were carried to 5μ; the effect falls off slowly and gradually from 3.5 to 5μ. In both holmium garnet and yttrium garnet[2] monotonic variation of the Faraday effect will evidently be observed in the long-wave portions of the dispersion curves at visible and ultraviolet characteristic frequencies.

If it is assumed that in the present case the Faraday effect consists of three terms due to three magnetic sublattices, by comparing the curves in Figs. 2 and 3 we can distinguish the rotation of the polarization plane caused by the rare-earth sublattice consisting of holmium ions. Since the Faraday effect in holmium garnet is smaller but of the same sign as for yttrium garnet, the rotation of the polarization plane due to the holmium sublattice is smaller and opposite to that caused by the magnetic iron sublattices.

We have detected a similar reduction of the infrared Faraday effect in erbium and gadolinium garnets. The Faraday effect in yttrium garnet falls off with decreasing temperature (Fig. 3); this is evidently due to narrowing of the absorption lines of iron ions at lower temperatures. In holmium garnet the Faraday effect is enhanced with decreasing temperature. This is associated either with the occurrence of electronic transitions at 1.96 and 1.14μ, or with a sharper temperature dependence of the Faraday effect for the holmium sublattice than for the iron sublattices.

We are indebted to V. A. Timofeeva, who grew the holmium garnet single crystals with the assistance of K. P. Belov.

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PRODUCTION OF Ξ^- HYPERONS BY 7- AND 8-Bev/c π^- MESONS

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The production and decay of Ξ^- hyperons produced in the interaction between π^- mesons in propane at momenta of 6.8 ± 0.6 and ~ 8 Bev/c were studied. A total of 11 Ξ^- hyperons were observed. Their decay energy Q (1), lifetime τ_0 (2), and production cross section (3) are given.

THE first cascade hyperons were obtained in cosmic-ray experiments. A brief review of the data on 16 Ξ^- hyperons produced by cosmic rays was given by Franzinetti and Morpurgo.[1] Later on, Ξ^- hyperons were produced in accelerators by 5.5-Bev/c π^- mesons (2 cases)[2] and by 1.15-Bev/c K^- mesons (16 cases).[3]

We used for our experiment a 24-liter propane bubble chamber placed in a constant magnetic field of 13 700 oe. The chamber was exposed to beams of π^- mesons of ~ 7 and ~ 8 Bev/c. The experimental arrangement has been described previously.[4] We scanned 27 000 photographs with 6.8 ± 0.6 Bev/c π^- mesons and 75 000 photographs with ~ 8-Bev/c π^- mesons. The photographs were scanned twice on stereo-viewers and reprojectors; part of them were scanned three times. In the scanning, we selected all events which could be ascribed by inspection to decays of cascade particles via the scheme $A \rightarrow V^0 + B$, $V^0 \rightarrow C + D$ (see, for example, Figs. 1 and 2) and also all secondary one-prong stars from whose vertex a V^0 particle is emitted. In this way we selected 90 cases.

The measurements were made on UIM-21 microscopes by measuring the coordinates of corresponding points on two stereo-photographs. The results of the measurements were fed to a Ural electronic computer, which gave the coordinates of the points, and the values of the momenta and angles. In the determination of the error in the particle momenta, we took into account the inaccuracy of the coordinate measurements on the microscope, the change in the track curvature due to multiple scattering in propane, and the nonuniformity of the magnetic field over the chamber volume.

The following criteria were used to identify the Ξ^- hyperons: 1) the character of the V^0 decay should be in agreement with the kinematics of the decay of Λ^0 particles into a proton and π^- meson; 2) the vertex formed by tracks A and B should lie in the decay plane of the Λ^0-particle decay, where the transverse momenta of the secondary π^- meson and proton should be balanced with respect to the direction of flight of the Λ^0 particle; 3) the point of decay of the Λ^0 particle should lie in the plane determined by the tracks of particles A and B; 4) the transverse momenta of the Λ^0 and B particles at the vertex formed by tracks A and B should be balanced; 5) the decay scheme of the Ξ^- decay, i.e., $\Xi^- \rightarrow \Lambda^0 + \pi^- + 65$ Mev, should be satisfied. Moreover, when possible, we checked the ionization of the decay products.

After analysis, we discarded 47 cases in which the V^0 decay was not associated with a vertex formed by A and B. Fourteen cases satisfied criterion 2, but the V^0 particles were identified as K_1^0. Of 29 cases satisfying criteria 1 and 2, only 15 satisfied criterion 3 (coplanar cases), but four of them did not satisfy criteria 4 and 5. Among the remaining 11 cases satisfying all five criteria, one was obtained in the 6.8 Bev/c π^--meson beam and 10 in the π^--meson beam at ~ 8 Bev/c. There were three cases (of the eleven) in which the V^0 particles were in agreement, within the limits of experimental error, with a K_1^0 or Λ^0 decay, where the ionization measurements did not make possible a choice between the Λ^0 and K^0 decays, owing to the large momenta of the positive decay products. However, since all three cases were in good agreement with the kinematics of a Ξ^- decay, we also attributed these cases to hyperons.

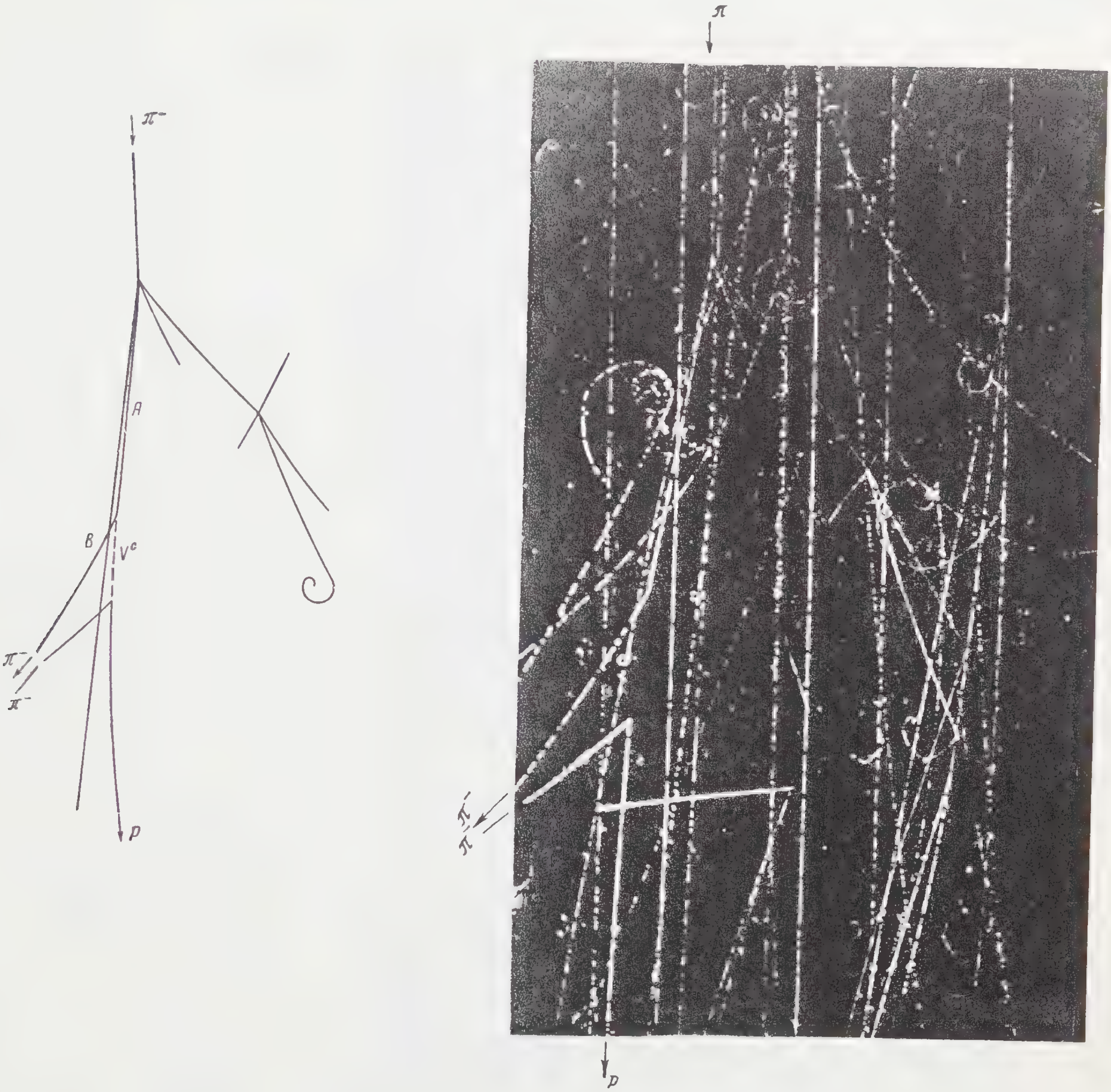

FIG. 1. Event No. 171-218. Decay of a Ξ^- hyperon.

Table I lists the angular and momentum characteristics of the Ξ^- hyperons identified by us. All the data were obtained by averaging the results from two to four independent measurements on the microscope. Also shown in the table are energy of decay Q and the lifetime up to the decay of the Ξ^- hyperons in their rest system.

The mean value of Q for the 11 cases of Ξ^--hyperon decay is

$$Q = 61{,}9 \pm 2{,}2 \text{ Mev} \tag{1}$$

from which we obtain the mass of the particle: $M_{\Xi^-} = 1317.0 \pm 2.2$ Mev. The lifetime of the Ξ^- hyperon was calculated by the maximum likelihood method and turned out to be

$$\tau_0 = \left(3.5 \begin{smallmatrix} +3.4 \\ -1.2 \end{smallmatrix}\right) \cdot 10^{-10} \text{ sec.} \tag{2}$$

After the identification of the Ξ^- hyperons, we analyzed the primary stars in which they were produced. Except for two cases (Nos. 6-230 and 370-252), all Ξ^- hyperons were produced inside the chamber. The tracks from the primary stars were measured twice and the momentum and energy balance was calculated for each star. The results are given in Table II. Six Ξ^- hyperons were produced in stars with an even number of prongs and resultant charge of zero. However, only five of them could be interpreted as π^-p interactions. (In event No. 171-218, the change in momentum in the collision was larger than the change in energy.)

Table I. Data on identified Ξ^- hyperons

p_{π^-}, Bev/c	Event No.	p_{Ξ^-}, Mev/c*	p_Λ, Mev/c	p_{π^-}, Mev/c	$\Theta_{\Xi^- -\Lambda}$	$\Theta_{\Xi^- -\pi^-}$	$\Delta p_\perp$, Mev/c**	Angle of noncoplanarity η***	Q_{Ξ^-}, Mev	τ_0, 10^{-10} sec
~7	182—42	3517 ± 420	3166 ± 415	374 ± 67	2°54′ ± 30′	18°30′ ± 30′	41 ± 42	9′	63.4 ± 14.4	1,13 ± 0,11
~8	91—145	1894 ± 150	1507 ± 150	406 ± 48	4°7′ ± 40′	15°25′ ± 40′	1 ± 25	14′	55,7 ± 13.7	2.53 ± 0.11
~8	196—160	1247 ± 185	963 ± 180	317 ± 44	6°45′ ± 30′	23°11′ ± 30′	12 ± 28	18′	69.1 ± 18.5	3,37 ± 0.27
~8	19—179	2811 ± 490	2407 ± 490	445 ± 64	4°9′ ± 30′	18°43′ ± 30′	31 ± 46	20′	73.2 ± 16.5	0.93 ± 0.11
~8	171—218	1398 ± 235	1181 ± 230	263 ± 29	6°22′ ± 1°20′	31°36′ ± 30′	8 ± 38	47′	64.4 ± 11.0	3.29 ± 0.37
~8	6—230	2008 ± 190	1583 ± 185	438 ± 39	2°58′ ± 1°	13°2′ ± 50′	17 ± 29	1°4′	52.5 ± 11.5	1.9
~8	370—252	878 ± 155	753 ± 151	197 ± 39	10°1′ ± 40′	46°20′ ± 30′	11 ± 39	2′	67.5 ± 20.3	6,4
~8	114—290	751 ± 93	625 ± 94	196 ± 19	13°1′ ± 27′	43°27′ ± 27′	6 ± 30	3° ****	63.4 ± 9.5	2,79 ± 0.41
~8	355—298	1812 ± 227	1593 ± 223	253 ± 44	4°58′ ± 1°30′	27°25′ ± 30′	21 ± 51	58′	53.2 ± 17.3	0.61 ± 0.09
~8	150—307	2440 ± 307	2098 ± 305	372 ± 45	5°1′ ± 1°	19°42′ ± 40′	58 ± 48	1°26′	56.2 ± 15,3	0.46 ± 0,07
~8	186—336	982 ± 115	702 ± 116	315 ± 51	8°35′ ± 30′	24°4′ ± 30′	23 ± 27	24′	86 ± 26	0.77 ± 0.14
Mean value		1794							61.9 ± 2.2	2.20

*Owing to the fact that some tracks were short, not all momenta could be obtained directly from the measurements. The values listed were calculated from the formula $p_{\Xi^-} = p_\Lambda \cos \Theta_{\Xi\Lambda} + p_\pi \cos \Theta_{\Xi\pi}$.

**$\Delta p_\perp = |p_\Lambda \sin \Theta_{\Xi\Lambda} - p_\pi \sin \Theta_{\Xi\pi}|$.

***η is the angle between the primary particle of the one-prong star and the plane determined by the Λ^0 particle and the secondary particle from the star; the error in the determination of the angle η is ~ 1°30′.

****The large error is due to difficulties of measurement.

Table II. Data on primary stars in which Ξ^- hyperons are produced

p_{π^-}, BeV/c	Event No.	Type of star	Other identified strange particles	$\sqrt{(\Delta E)^2+(\Delta p)^2}$, Mev *	Possible reaction of type**
~7	182—42	2-prong (1+, 1–)		1826 ± 210	(1)
~8	91—145	10-prong (7+, 2– and 1?)	K^+, stops and decays	394^{+150}_{-270}	(2)
~8	196—160	2-prong (1+, 1–)		1880 ± 180	(1)
~8	19—179	6-prong (3+, 3–)	K^+, stops and decays	185^{+100}_{-140}	(3)
~8	171—218	4-prong (2+, 2–)		$\Delta p > \Delta E$	(4)
~8	114—290	5-prong (4+, 1–)	K^0, decays	$\Delta p > \Delta E$	(5)
~8	355—298	3-prong (1+, 2–)		~1100	(6)
~8	150—307	2-prong (1+, 1–)		~2100	(7)
~8	186—336	2-prong (1+, 1–)		~1520	(1)

*The value of ΔE was calculated under the assumption of an interaction between a π^- meson and a free nucleon.

** $\pi^- + p \rightarrow \Xi^- + \pi^+ + (K^0 + K^0 + \ldots)$ (1)

$\pi^- + C \rightarrow \Xi^- + K^+ + K^+ + \pi^+ + 4p + \pi^- + \ldots$ (2)

$\pi^- + p \rightarrow \Xi^- + K^+ + K^+ + \pi^+ + \pi^- + \pi^- + \pi^0$ (3)

$\pi^- + C \rightarrow \Xi^- +$ other particles (4)

$$\pi^- + C \left\} \begin{array}{l} \rightarrow \Xi^- + K^0 + \pi^+ + \pi^+ + 2p + (K^0 + \ldots) \\ \rightarrow \Xi^- + K^0 + K^+ + \pi^+ + 2p \ldots \end{array} \right. \quad (5)$$

$$\pi^- + n \left\} \begin{array}{l} \rightarrow \Xi^- + \pi^- + \pi^+ + (K^0 + K^0) \\ \rightarrow \Xi^- + \pi^- + K^+ + (K^0 + \ldots) \end{array} \right. \quad (6)$$

$$\pi^- + p \left\} \begin{array}{l} \rightarrow \Xi^- + \pi^+ + (K^0 + K^0 + \ldots) \\ \rightarrow \Xi^- + K^+ + (K^0 + \ldots) \end{array} \right. \quad (7)$$

Assumed particles are shown in parentheses.

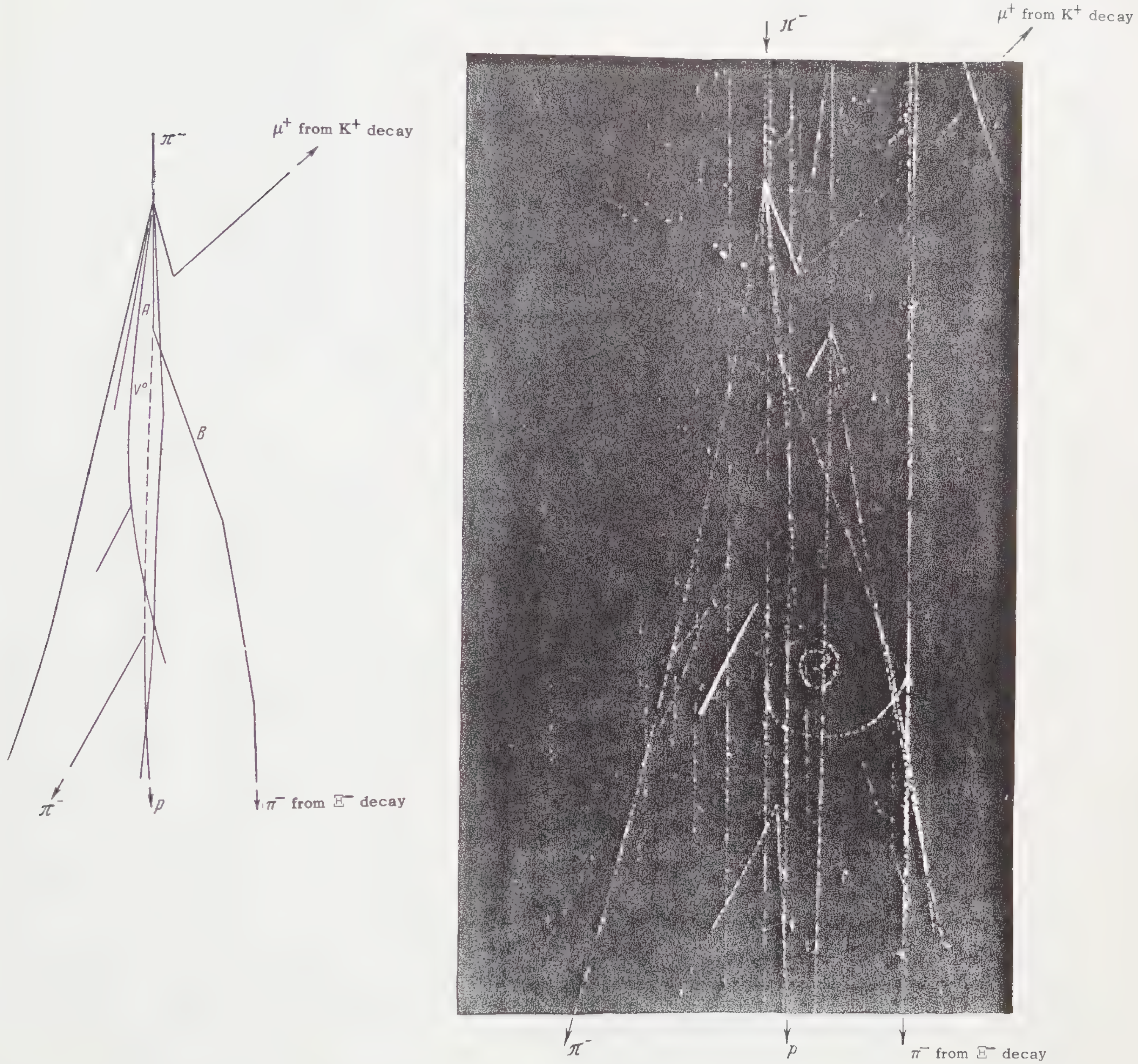

FIG. 2. Event No. 19–179. Decay of a Ξ^- hyperon.

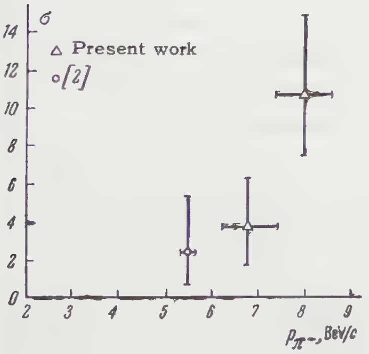

FIG. 3. Dependence of Ξ^--hyperon production cross section on energy of incident π^- mesons.

Assuming a strangeness of $S = -2$ for the Ξ^- particle, we should expect that two K mesons (K^0, K^+) with $S = +1$ are produced along with the cascade hyperon. Only in three cases could K mesons be identified from the kinematics of the decay. In event No. 91-145, four stopping protons were identified; a π^+ meson was well-identified from its momentum and ionization and a K^+ meson, from the kinematics of the decay. The seventh particle, which was positive, could not be a π^+ meson because of its momentum and ionization; this means that it was either a K^+ meson or a proton. In event

Table III.

p_{π}, Bev/c	Event No.	$p^*_{\Xi^-}$ (in π—N c.m.s.), Mev/c	$(p_\perp)_{\Xi^-}$, Mev/c	$\theta^*_{\Xi^-}$ (in π—N c.m.s.), deg	θ^*_{Λ} (in Λ^0 system), deg	θ^*_p (in Λ^0 system), deg	$\overset{0}{\omega}_{\Xi\Lambda}$, angle between decay planes of Ξ^- and Λ^0, deg	Direction of flight of proton relative to the Ξ^--decay plane
~7	182—42	558 ± 62	375 ± 48	42 ± 11	71 ± 5	92 ± 5	146 ± 5	down
~8	91—145	414 ± 47	145 ± 26	159 ± 4	122 ± 5	42 ± 5	131 ± 4	up
~8	196—160	1001 ± 99	290 ± 30	162 ± 3	123 ± 5	67 ± 6	52 ± 3	down
~8	19—179	294 ± 41	293 ± 41	94 ± 11	88 ± 10	73 ± 5	161 ± 5	down
~8	171—218	977 ± 107	410 ± 47	155 ± 4	94 ± 5	50 ± 7	87 ± 4	down
~8	6—230				132 ± 8	75 ± 5	100 ± 6	up
~8	370—252				91 ± 10	130 ± 5	23 ± 5	up
~8	114—290	1538 ± 110	273 ± 36	170 ± 3	102 ± 5	105 ± 5	168 ± 6	up
~8	355—298	1070 ± 102	729 ± 92	137 ± 6	73 ± 5	35 ± 10	82 ± 4	down
~8	150—307	140^{+145}_{-64}	71 ± 31	149^{+22}_{-86}	75 ± 15	70 ± 15	88 ± 8	down
~8	186—336	1248 ± 110	273 ± 36	167 ± 4	126 ± 4	102 ± 5	66 ± 4	down
Mean value		804	318 ± 35					

Table IV. "Background" cases (the V^0 particles were identified as Λ particles)

Event No.	Sign of charge	Angle of noncoplanarity η	$\Delta p_\perp$, Mev/c	One of the possible interactions*	Event No.	Sign of charge	Angle of noncoplanarity η	$\Delta p_\perp$, Mev/c	One of the possible interactions*
344—314	+	1′	225	(1)	273—125	+	10°10′	827	(5)
250—35	+	16′	200	(1)	140—45	+	22°42′	244	(3) ***
267—233	+	1°19′	48	(1)	99—68	—	3°36′	86	(2)
150—286	—	22′	126	(2)	179—221	—	4°27′	125	(6) **
502—12	+	2°39′	56	(1)	144—219	—	7°4′	106	(7)
185—91	+	4°56′	58	(3)	152—188	—	16°40′	59	(6) **
52—246	+	4°57′	16	(1)	64—69	—	17°30′	70	(6) **
189—275	+	5°54′	17	(1)	61—174	—	30°		(8)
33—221	+	8°53′	497	(4)	288—193	—	40°	50	(6) **

* $\pi^+ + n \rightarrow \Lambda + K^+$ (1)
$K^- + n \rightarrow \Lambda + \pi^-$ (2)
$\pi^+ + n \rightarrow \Lambda + (K^0) + \pi^+$ (3)
$\pi^+ + C \rightarrow \Lambda + p + (K^0) + \ldots$ (4)
$\pi^+ + n \rightarrow \Lambda + (+?) + \ldots$ (5)
$\pi^- + n \rightarrow \Lambda + \pi^- + \ldots$ (6)
$K^- + n \rightarrow \Lambda + \pi^- + \ldots$ (7)
$\pi^- + n \rightarrow \Lambda + \pi^- + K^0 + \ldots$ (8)

**Possible $K^- + n$ interaction.

***In this case another reaction with the production of a Λ particle is possible; we have indicated the more probable one.

No. 19-179, under the assumption of a π^-p interaction, the neutral particle could be only a π^0 meson, and then the positive particle, which could not be identified from its momentum and ionization, should be a K^+ meson. Hence there was not a single case of associated production of a Ξ^- hyperon with two well-identified K^+, K^0 mesons, but neither were there any cases which would contradict such a scheme of associated production.

Table III lists the momentum p*, the transverse momentum $p_\perp$, and the angle of flight θ^* of the Ξ^- hyperons in the c.m.s. of the π^- meson and nucleon under the assumption that the Ξ^- particle is produced in a collision of primary π^- mesons with free nucleons.

As seen from Table III, the mean transverse momentum $\overline{p}_\perp$ is equal to 318 ± 35 Mev/c. It is interesting to note that this value of $\overline{p}_\perp$ for the Ξ^- hyperons is close to the value of the transverse momentum for protons and Λ^0 hyperons.[5] Moreover, the Ξ^- hyperons are emitted mainly backwards, similarly to protons and Λ^0 and $\Sigma^\pm$ hyperons in π^-p interactions involving 6.8-Bev/c mesons.[6]*

Table III also lists the following angular characteristics: the angle of flight of the Λ^0 particle in the Ξ^- rest system θ^*_Λ, the angle of flight of the proton from the Λ^0-particle decay in the Λ^0 rest system θ_p^*, and the angle $\omega_{\Xi\Lambda}$ between the Ξ^-- and Λ^0-particle decay planes. No asymmetry in the θ_Λ^* and $\omega_{\Xi\Lambda}$ distributions was observed.

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*It should be noted that both Ξ^- hyperons observed in reference 2 are emitted backwards in the c.m.s.

In the forward-back (θ_p^*) and up-down distributions of the proton decay products from the Λ^0-particle decay, some asymmetry was observed (seven cases forward, three cases backwards, one at an angle of 90°; eight cases down, three cases up). We note that the protons from the Λ^0 decays are directed forward in the Λ^0 rest system.[2] An asymmetry in the angular distributions characterizes the polarization of the Λ^0 particles from the Ξ^--particle decay. Longitudinal polarization of the Λ^0 hyperons from the decay of Ξ^- particles, if it exists, would be evidence of the parity nonconservation in the decay of Ξ^- hyperons.

In order to determine the production cross section for Ξ^- hyperons, we counted the number of π^- mesons in the primary beam and determined the effective length of the chamber for the recording of Ξ^- hyperons; this was equal to 30 ± 5 cm. We then took into account corrections for the following: 1) the μ^--meson contamination $(5 \pm 2)\%$; 2) the loss of part of the primary π^- mesons due to interactions in the chamber;* 3) the efficiency of finding Ξ^- hyperons in scanning the photographs $(80-90\%)$; 4) the geometric conditions for recording the particles (2.0 ± 0.3); and 5) the decay of secondary Λ^0 hyperons into neutral particles. We thus found the mean free path of π^- mesons for the production of Ξ^- hyperons in propane to be $l = (2.02^{+2.86}_{-0.84}) \times 10^6$ cm for 6.8-Bev π^- mesons and $l = (0.68^{+0.29}_{-0.20}) \times 10^6$ cm for π^- mesons at ~ 8 Bev/c. If it is assumed that the production cross section for hyperons on nuclei is proportional to $A^{2/3}$, then for the Ξ^--hyperon production cross section (per nucleon) in propane, we have

$$\sigma = 3{,}6^{+2.5}_{-2.1}\ \mu\text{b} \quad \text{at} \quad 6.8 \quad \text{Bev/c}$$

$$\sigma = 10{,}6^{+4.4}_{-3.2}\ \mu\text{b} \quad \text{at} \quad \sim 8\ \text{Bev/c}\,. \tag{3}$$

In the work of Fowler et al.[2] the cross section for the production of Ξ^- hyperons (per nucleon) by 5.5-Bev/c π^- mesons was found to be $\sigma = 2.3^{+3.1}_{-1.6}\,\mu$b. Figure 3 shows the variation of the Ξ^- hyperon production cross section as a function of the incident π^- meson energy.† Finally, Table IV lists the "background" cases, among which are four coplanar and 14 noncoplanar cases. The coplanar cases can be explained well by the reactions indicated in Table IV. Since the reaction $\pi^- + n \rightarrow \Lambda^0 + K^-$ is forbidden by the law of conservation of strangeness and since there are less K mesons than π^+ mesons, the greater part of the coplanar cases are due to π^+ mesons. The noncoplanar cases can be caused by both π^+ and π^- or K^- mesons. In our case, there were seven interactions caused by π^+ mesons and seven caused by negative particles. Interactions Nos. 150-286, 99-68, and 144-219 could be due only to K^- mesons, since the energy of the π^- meson in these cases was below the threshold for the production of a Λ^0 particle.

In conclusion, the authors express their gratitude to V. I. Veksler and I. V. Chuvilo for discussion of the results, to L. P. Zinov'ev, N. I. Pavlov, K. V. Chekhlov, L. N. Belyaev, the engineers and technicians of the accelerator group, and the technicians and laboratory personnel of the bubble chamber group for aid in carrying out the experiment, to the laboratory staff for performing the measurements, and to the electronic computer group for carrying out the calculations.

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*The interaction mean free path of 6–8 Bev/c π^- mesons in propane is equal to 219 ± 5 cm.

†We recall that the threshold for the production of hyperons by π^- mesons via the reaction $\pi^- + N \rightarrow \Xi^- + K + K$ with allowance for the motion of the nucleons in the nucleus is equal to $2.19^{+0.16}_{-0.43}$ Bev.[7]

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ATOMIC FIRST IONIZATION POTENTIALS DETERMINED BY THE METHOD OF SURFACE IONIZATION

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Atoms of the rare-earth metals Er, Tb, and Ce, together with the molecule $ThCl_4$, have been studied using the method of surface ionization on polycrystalline tungsten. It is shown that the temperature dependence of the ion current agrees with the theory of surface ionization on complex surfaces. The ionization potentials of the atoms Er, Tb, Ce, and Th have been found by comparing the temperature dependence of their respective ion currents with the same quantity for positive In ions.

THE method previously described[1,2] has been used to study the ionization of several rare earth metals on a tungsten surface. In Fig. 1 the logarithm of the relative positive ion current log (I/I_m) for the atoms of Er, Tb, Ce and In is plotted against the inverse temperature, $1/T$, of the tungsten wire. At sufficiently high temperatures the graph is a straight line, as predicted by the theory.[1,3]

The ionization potential V of an atom can be found by comparing the temperature dependence of the positive ion current for the unknown atom with the same quantity for a known atom. This method has been used in work reported earlier to obtain V for atoms of U,[1] Nd, and Pr.[2] Our measurements were made relative to In, whose ionization potential is known spectroscopically to be $V_{In} = 5.79$ ev.

Let I_1 denote the current of positive In ions, and I_2 the current of positive ions of Er, Tb or Ce, as the case may be. In Fig. 2, the logarithm of the ratio I_1/I_2 is plotted against the inverse temperature $1/T$.[1,3] The three curves are straight lines to a good approximation. The slopes can be used to obtain the differences between the ionization potentials of the three elements and that of indium. Eight independent measurements were made of V for Er, nine for Tb and five for Ce. The results are as follows:

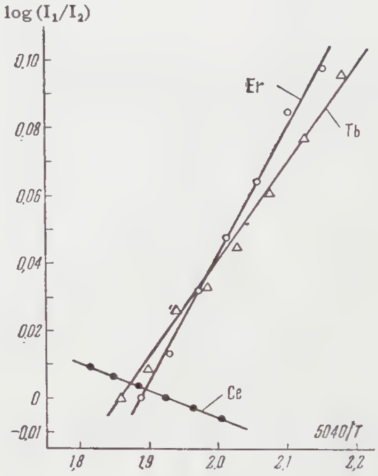

FIG. 2

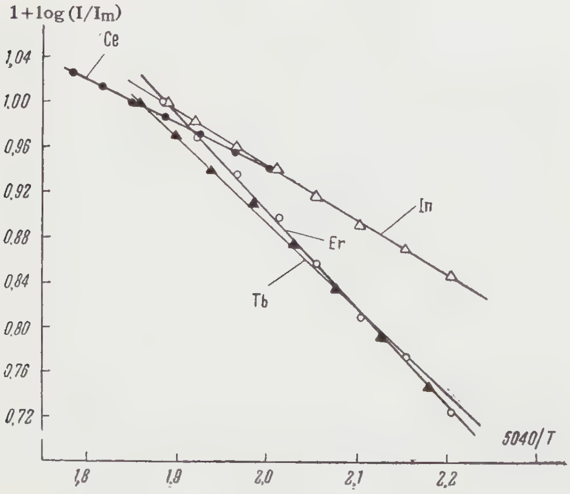

FIG. 1

$$V_{Er} = 6{,}08 \pm 0{,}03 \text{ ev}, \qquad V_{Tb} = 5{,}98 \pm 0{,}02 \text{ ev}, \qquad V_{Ce} = 5{,}60 \pm 0{,}05 \text{ ev}.$$

The ionization potential of Th was obtained in a similar manner, molecules of anhydrous $ThCl_4$ being ionized on a tungsten wire together with atoms of In. The temperature of the wire was 2500 to 2900° K. The ionization potential of Th turned out to be 6.95 ± 0.06 ev.

The calculations took into account those excited states of In atoms and ions which lie close to the ground states. Excited states in the atoms and ions of Er, Tb, Ce and Th were not taken into account since the energy levels for these elements are not known.

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AN INVESTIGATION OF (n, 2n) *REACTIONS LEADING TO ISOMER FORMATION*

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Data are presented on the cross sections for (n, 2n) reactions involving 14.7-Mev neutrons and resulting in the formation of the isomers Y^{88m} ($\sigma_m > 0.4$ barn), Nb^{92g} ($\sigma_g = 0.53 \pm 0.06$ barn), Pb^{207m} ($\sigma_m = 1.7 \pm 0.3$ barn), and Bi^{208m} ($\sigma_m = 0.66 \pm 0.12$ barn). The cross section for the reaction with lead was evaluated on the assumption that the isomer is produced only in the reaction $Pb^{208}(n, 2n)Pb^{207m}$. This value is compared with that computed from strong-interaction theory, assuming a two-stage mechanism of neutron evaporation in the (n, 2n) reaction. The Pb^{207} level scheme which agrees with the shell model was employed in the calculations. The relative probabilities of different types of transitions were estimated from the relation between the lifetime of an excited nuclear state and the transition energy. There is good agreement between the measurements and calculations. Radiation from the following isomers is reported: Nb^{92g} ($T_{1/2} = 10.0 \pm 0.3$ day, $E_\gamma = 0.94 \pm 0.01$ Mev), Bi^{208m} ($T_{1/2} = 2.6 \pm 0.1$ millisec, $E_\gamma = 0.50 \pm 0.02$ Mev, $E_\gamma = 0.88 \pm 0.02$ Mev), and Na^{24m} ($T_{1/2} = 18.3 \pm 0.6$ millisec).

PROCEDURE

OUR experimental technique and preliminary estimates of the cross sections for reactions with fast neutrons resulting in the formation of isomers with millisecond half-lives are similar to those described in reference 1. We have also published[2] an account of certain changes in technique together with the cross section obtained for the reaction producing Pb^{205m}. The technique described in reference 2 was used to measure the cross sections for the reactions producing Pb^{207m} and Bi^{208m}. The present paper gives the results of these measurements and data for additional isomers.

EXPERIMENTS

Samples with natural isotopic content were irradiated with neutrons produced in the pulsed bombardment of a thick tritium-zirconium target by D_2^+ ions from a 500-kv accelerating tube. The neutron spectrum was computed; the neutrons interacting with Pb, Nb, and Bi had a peak at $E_0 = 14.7$ Mev and a 0.25-Mev half-width.

γ rays from the samples induced pulses in a FÉU-S photomultiplier used in conjunction with a NaI(Tl) crystal; these pulses were then fed to amplitude and time analyzers during the intervals between neutron pulses. The cross section for a given reaction was determined by counting the pulses forming the photopeak of total isomeric emission in the pulse-height distribution.

The efficiency of γ-ray registration by means of the photopeak pulse counts was determined with radioactive Cs^{137}, Nb^{95}, Nb^{92}, and Zn^{65} samples, whose activities were measured by two methods that produced mutually agreeing results. One method, as in reference 1, employed a counter with a special sheath, described by Sakharov;[3] the other method used a NaI(Tl) crystal and spectrometer. In the latter case the efficiency of γ-ray registration was computed with the aid of data given in references 4 and 5. We estimated a 7 to 8% inaccuracy in determining the activities.

The yield of a given reaction as determined by the monitor was compared with the yield of $Cu^{63}(n, 2n)Cu^{62}$, for which the mean cross section 0.62 ± 0.04 barn was obtained from the computed neutron spectrum. The cross section for the investigated reaction was then calculated from the formula given in reference 1.

RESULTS

Our data give a half-life of 0.81 ± 0.02 sec for Pb^{207m}, in agreement with the review article of Strominger et al.[6] The cross section for the reaction with lead was determined using a 1.04-Mev photopeak, for which pulses falling in 0.768 ± 0.006 sec intervals were discriminated. The start of these intervals was delayed for the time t (0.09

$\leq t \leq 0.11$ sec) after the leading edge of the ion pulse of the accelerating tube.

In interactions between 14-Mev neutrons and heavy nuclei a (n, 2n) reaction is the dominant inelastic process. For lead, bismuth, and certain other elements this has been confirmed by Ashby et al.,[7] who reported a cross section for Pb (n, 2n) equal to the cross section for all inelastic processes: $\sigma_{(n,2n)}/\sigma_{(n,x)} = 1.07 \pm 0.08$. We assumed that in our experiments Pb^{207m} was formed only in a (n, 2n) reaction, and attributed our results to Pb^{208}. We obtained $\sigma_m^{meas} = 1.7 \pm 0.3$ barns for the $Pb^{208}(n, 2n)Pb^{207m}$ reaction, and compared this result with the calculated cross section σ_m^{calc}.

The cross section for $Bi^{209}(n, 2n)Bi^{208m}$ was determined similarly. γ rays with 0.88 ± 0.02 Mev and 0.50 ± 0.02 Mev, and a half-life of 2.6 ± 0.1 millisec, were observed in the decay of Bi^{208m}. The pulse count gave a 0.88-Mev photopeak and cross section $\sigma_m = 0.66 \pm 0.12$ barn.

We also measured the cross section for the reaction producing a long-lived niobium isomer, and improved the half-life of Na^{24m}. The Nb^{92g} samples used to determine the efficiency of γ-ray registration were prepared by neutron bombardment of thin metallic niobium plates. We obtained 10.0 ± 0.3 days for the half-life of this isomer and detected 0.94 ± 0.1 Mev γ rays accompanying β decay. The experiments in which the reaction yield was determined from the radioactivity of the sample have already been discussed. We used the decay schemes of Nb^{92m} and Nb^{92g} given by Dzhelepov and Peker.[8] The cross section for $Nb^{93}(n, 2n)$ Nb^{92g} was $\sigma_g = 0.56 \pm 0.06$ barn.

Na^{24m} was formed in the reaction $Al^{27}(n, \alpha)$ Na^{24m}. γ rays from the isomer were registered by a spectrometer used with a NaI(Tl) crystal whose packing contained neither aluminum nor magnesium. There was no background which could have been due to the reaction $Na^{23}(n, \gamma)$ Na^{24m}. An improved value of 18.3 ± 0.6 msec was obtained for the half-life of Na^{24m} given in reference 1.

From earlier data[1] obtained with 14.9-Mev neutrons we estimated the cross section for the reaction producing the isomer with 14-millisec half-life in the fast-neutron bombardment of yttrium. The result $\sigma_m > 0.4$ barn provided a sufficient basis for assuming that the given isomer resulted from a (n, 2n) reaction and for attributing the 0.24-Mev emission to Y^{88m}.[1]

CALCULATION OF THE CROSS SECTION FOR $Pb^{208}(n, 2n)Pb^{207m}$

The cross section for the reaction producing Pb^{207m} was calculated on the basis of the strong-interaction theory and two-stage neutron evaporation. We calculated the partial cross sections for the formation of $Pb^{208} + n$ compound nuclei in states with spin j, as follows:

$$\sigma(j) = \pi\lambda^2(2j+1) \sum_{s=I_0-1/2}^{I_0+1/2} \sum_{l=|j-s|}^{j+s} T_l(E_0)/2(2I_0+1), \quad (1)$$

where $T_l(E)$ is the centrifugal barrier transmission coefficient for neutrons with orbital angular momentum l and energy E, $I_0 = 0$ is the spin of Pb^{208}, and $1/\lambdabar = k$ is the wave number of a neutron with energy E_0. We then calculated the partial cross sections for the formation of Pb^{208*} in states with spin j_1 following the evaporation of the first neutron:

$$\sigma(j_1) = \sum_j \sigma(j)\Big[\sum_{s_1=j_1-1/2}^{j_1+1/2} \sum_{l_1=|j-s_1|}^{j+s_1} \int_0^{E'_{max}} T_{l_1}(E') f(E')\,dE'\Big]$$
$$\times \Big[\sum_{j_q} \sum_{s_q=j_q-1/2}^{j_q+1/2} \sum_{l_q=|j-s_q|}^{j+s_q} \int_0^{E'_{max}} T_{l_q}(E') f(E')\,dE'\Big]^{-1}, \quad (2)$$

where $f(E')dE' = \text{const} \times E' \exp(-E'/\tau_1)dE'$ is the energy distribution of neutrons evaporating in the first stage, at the temperature $\tau_1 = 1$ Mev.

Equations (1) and (2) were derived from the equation of Hauser and Feshbach[9] that gives the cross section for the excitation of an individual level in inelastic neutron scattering. We separated the parts pertaining to the formation and decay of the compound nucleus, and integrated over the energies of evaporating neutrons instead of summing over the energies of separate levels. It was assumed that compound nuclei decay into close energy levels. In Eq. (2) the spin dependence of level density was taken into account by the summations over $l_{q(1)}$ and $s_{q(1)}$, and the dependence on excitation energy was taken into account by the function $f(E')$.

We kept in mind the remark of Remy and Winter[10] that τ_1 must be 20 — 30% above the temperature determined experimentally (0.7 — 0.8 Mev according to references 11 and 12), neglecting two-stage evaporation. We also assumed that τ_1 can differ little from the result 1.05 Mev obtained by Rosen and Stewart for the Bi (n, 2n) reaction.[13]

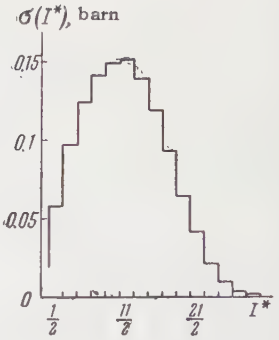
It can be assumed that with the evaporation of a second neutron Pb^{208*} compound nuclei decay to states of Pb^{207*}, where the level density is sufficient to permit the use of a near-Maxwellian energy distribution in the calculations. Neutrons in the range 5 — 12 Mev were not observed in the case of the reactions with lead reported in references 11 and 12. These neutron energies would have been observed if decay to discrete levels were dominant. There are both small and large spin values among the low-energy levels of Pb^{207}. The probability of decay to these levels, which contains the factor $T_l(E')$, would be relatively large.

Keeping the foregoing considerations in mind, we used (2) to calculate the spin distribution of Pb^{207*} nuclei formed after the evaporation of a second neutron. j_1 and j were replaced by I* and j_1, respectively, and the energy distribution was characterized by the temperature $\tau_2 = 0.5$ Mev, which is close to the values 0.4_6 and 0.5_3 Mev estimated by Rosen and Stewart[13] for the Ta (n, 2n) and Bi (n, 2n) reactions.

Penetration of the centrifugal barrier was calculated from

$$\begin{aligned}[T_l(x)]^{-1} = {}^1/_2 + \frac{\pi}{8X}\Big\{&(X^2 + l^2)[J^2_{-(l+1/2)}(x) + J^2_{l+1/2}(x)] \\ &+ x^2[J^2_{-(l-1/2)}(x) + J^2_{l-1/2}(x)] \\ &+ 2lx\,[J_{-(l+1/2)}(x)J_{-(l-1/2)}(x) \\ &- J_{l+1/2}(x)\,J_{l-1/2}(x)]\Big\}, \qquad (3)\end{aligned}$$

where $x = kR = 0.222\sqrt{E\,(\mathrm{Mev})} \times 1.45\,A^{1/3}$, $X^2 = x^2 + X_0^2$, and $J_{\pm(l+1/2)}(x)$ is the Bessel function of the first kind and of order $\pm(l \pm 1/2)$. X_0^2 is expressed in terms of the depth V_0 and radius R of the rectangular potential well representing the potential in the simplest type of strong interaction; thus $X_0^2 = 2mV_0R^2/\hbar^2$. Equation (3) was obtained from the book by Blatt and Weisskopf,[14] with $v_l(x)$ and $v'_l(x)$ replaced by Bessel functions. We used the value $X_0^2 = 144$, with which Oleksa[15] obtained best agreement between the calculated cross section for $Pb^{207}(n, n')Pb^{207m}$ and experimental data on inelastic neutron scattering at $1.6 \le E_0 \le 2.5$ Mev. It was estimated that with $X_0^2 = 64$, $\sigma(I^*)/\Sigma_{I^*}\sigma(I^*)$ can exceed the result obtained with $X_0^2 = 144$ by no more than 5%. The results obtained for $\sigma(I^*)$ are shown in the figure. Identical partial cross sections are obtained for the formation of Pb^{207*} in states with different parities.



Partial cross sections for the formation of Pb^{207*} nuclei in states with the same parity.

In view of the results obtained by Ashby et al.[7] we used

$$\sigma^{\mathrm{calc}}_{(n,2n)} = \sum_j \sigma(j) = \sum_{I^*} \sigma(I^*), \qquad \sigma^{\mathrm{calc}}_m = \sum_{I^*} r_m(I^*)\,\sigma(I^*)$$

to represent the cross sections for $Pb^{208}(n, 2n)Pb^{207*}$ and $Pb^{208}(n, 2n)Pb^{207m}$, respectively. $r_m(I^*)$ and $r_g(I^*)$ are the respective probabilities that a nucleus in a highly excited state with spin I* drops to a metastable state $(13/2^+)$ or directly to the ground state $[r_m(I^*) + r_g(I^*) = 1]$. These probability coefficients were evaluated from Montalbetti's nomogram[16] relating the mean lifetime of an excited nuclear state with the type and energy of a single-particle transition accompanied by γ-ray emission. It was assumed that this nomogram is applicable to the relative probabilities of different transition types independently of the nuclear model. It was also considered that from states with $E^* \approx 6.5 - 7$ Mev, including possible non-single-particle states, transitions proceed to single-particle Pb^{207} levels determined from the study of radioactive decay, given by Dzhelepov and Peker,[8] and proposed by Harvey[17] from an investigation of (d, p) and (d, t) reactions. The arrangement of these levels agrees with the shell model, as follows: $p_{1/2}$ (ground state); $f_{5/2}$, 0.570 Mev; $p_{3/2}$, 0.894 Mev; $i_{13/2}$, 1.633 Mev (metastable state); $f_{7/2}$, 2.34 Mev; $g_{9/2}$, 2.75 Mev; $i_{11/2}$, 3.60 Mev; $d_{5/2}$, 4.42 Mev; $g_{7/2}$, 4.66 Mev; $d_{3/2}$, 5.28 Mev. We allowed the existence of a $15/2^-$ level at ~ 5 Mev, assumed to result from the strong splitting of a level with orbital moment 7 due to spin-orbit interaction.

The values obtained for $r_m(I^*)$ by the described procedure are as follows. For all even states with spins $I^* \ge 11/2$ and odd states with spins $I^* \ge 7/2$, $r_m(I^*) = 1$. For all other states forming the distribution shown in the figure, $r_m(I^*) = 0$. It was here assumed that the probabilities of different types (E1, M1, E2, ...) of transitions were comparable if they did not differ by a factor greater than 20. For $E^* \gtrsim 6$ Mev there was no competition between transitions of the different types, and $r_m(I^*)$ was independent of E*.

It was to be expected that the coefficients $r_m(I^*)$ would be close to unity or zero and independent of the excitation energy. Katz and others[18-20] investigated the relative yields σ_g/σ_m for several isomers formed in (γ, n) reactions and found σ_g/σ_m to be practically constant over broad intervals of γ-ray energy. Dipole absorption predominated; it could therefore be assumed that the spin distributions of excited nuclei were only slightly dependent on energy, and the constant value of σ_g/σ_m could be accounted for by the fact that $r_{m(g)}(I^*)$ is independent of E^*. The summary of data in reference 21 gives the relative yields of isomers resulting from thermal neutron absorption by even-even nuclei; in the great majority of cases $r_{m(g)}(\frac{1}{2}^+)$ is close to unity. $r_{m(g)}(I^*) \approx 1$ from the results of experiments in which isomers were formed in (n, γ) reactions with thermal neutrons and with fission neutrons.[22-24]

The given values of $r_m(I^*)$ yielded $\sigma_m^{calc} = 1.58$ barns, which is in satisfactory agreement with σ_m^{meas}. We also obtained $(\sigma_m/\sigma_{n,2n})_{calc} = 0.65$; considering that this is only slightly dependent on X_0^2, we calculated the cross section $\sigma^0_{(n,2n)} = \sigma_m^{meas}(\sigma_{(n,2n)}/\sigma_m)_{calc} = 2.6$ barns for $Pb^{208}(n, 2n)Pb^{207*}$. Within the limits of experimental and calculational accuracy this result agrees with $\sigma_{(n,2n)} = 2.74 \pm 0.20$ barns in reference 7, and with the cross sections for inelastic interactions between neutrons with 14.2 — 14.5 Mev and lead nuclei, for which the results $\sigma_{(n,x)} = 2.56 \pm 0.03$ barns and 2.54 ± 0.05 barns are given in reference 25 and 26, respectively.

Natural lead was used in references 7, 25, and 26. Our results therefore permit application of the strong-interaction model and the roughly approximate statistical theory to the $(n, 2n)$ reaction in the case of the double magic isotope Pb^{208}. The agreement of $\sigma^0_{(n,2n)}$ with $\sigma_{(n,2n)}$ and $\sigma_{(n,x)}$, and of σ_m^{calc} with σ_m^{meas}, show that in analyzing the spin distributions of excited nuclei formed in reactions it is sometimes permissible to use the aforedescribed procedure for calculating the coefficients $r_m(I^*)$.

It has been shown in references 27 and 28 that the correlations of proton and neutron positions in nuclei can result in a very low probability of E1 transitions in the case of low excitation energies. This evidently applies to Pb^{207}, where the $g_{9/2} \rightarrow i_{13/2}$ transition, for example, has considerably higher probability than the $g_{9/2} \rightarrow f_{7/2}$ transition. We therefore repeated our calculation excluding electric dipole transitions in evaluating $r_m(I^*)$. We obtained $r_m(I^*) = 1$ for all odd states with spins $I^* \geq \frac{13}{2}$ and even states with $I^* \geq \frac{11}{2}$. For $I^* = \frac{3}{2}^+$ and $\frac{7}{2}^+$ competition is possible between M1 transitions (with relatively small transition energies) and M2 transitions (with transition energies ~ 6 Mev). For $I^* = \frac{3}{2}^+$ and $\frac{7}{2}^+$ we have $r_m(I^*) = 0.2$. For $I^* = \frac{9}{2}^+$ the M1, E2, and M2 transition probabilities are comparable and $r_m(\frac{9}{2}^+) = 0.9$. For the other states represented in the spin distribution shown in the figure we have $r_m(I^*) = 0$. The final results are $\sigma_m^{calc} = 1.32$ barns and $\sigma^0_{(n,2n)} = 3.1$ barns. These values agree with the values of σ_m^{meas}, $\sigma_{(n,2n)}$, and $\sigma_{(n,x)}$ given above. However, since the agreement was worsened when E1 transitions were excluded we consider it highly probable that when $E^* \gtrsim 3.6$ Mev in Pb^{207}, electric dipole transitions play an important part.

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THE g-*FACTORS FOR COLLECTIVE AND INTERNAL MOTION IN* Tb^{159} *AND* Yb^{173} *NUCLEI*

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The transition probabilities for magnetic dipole transitions from the first excited states in Tb^{159} and Yb^{173} nuclei have been measured with a multi-channel time analyzer. The gyromagnetic ratios calculated on the basis of the unified model are compared with results of experiments on Coulomb excitation. The experimental results show that the g-factors for collective motion are close to those estimated on the basis of the unified model according to the formula $g_R \approx Z/A$.

THE knowledge of the lifetimes of magnetic dipole transitions and in addition the magnetic moments μ of the nuclei allows one to determine separately the g-factors of collective motion (g_R) and internal particle motion (g_K) on the basis of the unified model.[1]

In the method of Coulomb excitation the reduced probability of E2 transitions B(E2) is determined directly from experiment, and the reduced probability of M1 transitions B(M1) is calculated from the ratio of the intensities of the two components $\delta^2 = \lambda(E2)/\lambda(M1)$, where λ is the transition probability. However, in those cases in which the M1 component makes up the preponderant fraction of the radiation, the determination of g-factors by the method of Coulomb excitation becomes very unreliable, since it is based on the accuracy of the knowledge of the small amount of E2 admixture.

It is clear that in such cases, which are frequent, it is expedient to make a direct measurement of the lifetime of the level by the method of delayed coincidences.

In Figs. 1 and 2 are shown the results of measuring the half-lives of the 58-kev level in Tb^{159} and the 78.7-kev level in Yb^{173}, obtained with a multichannel time analyzer which has been described earlier.[2]

In the case of Yb^{173}, coincidences of the 272-kev gamma ray with conversion electrons from the 78.7 kev transition were counted. A compound of Co^{60} was used as a comparison source (for β-γ coincidences). In the case of Tb^{159} coincidences of the characteristic x-rays with conversion electrons from the 58-kev level were counted, and the comparison source used was Hg^{203} (for β-γ coincidences). This comparison source exhibits the accurately known lifetime of the 279-kev level of Tl^{203}, and the weighted average value from many measurements[2,3] of the half-life is $T_{1/2} = (2.90 \pm 0.12) \times 10^{-10}$ sec. In this case, the use of Co^{60} as a comparison standard was inconvenient because of the excessive difference in the energies of the two radiations.

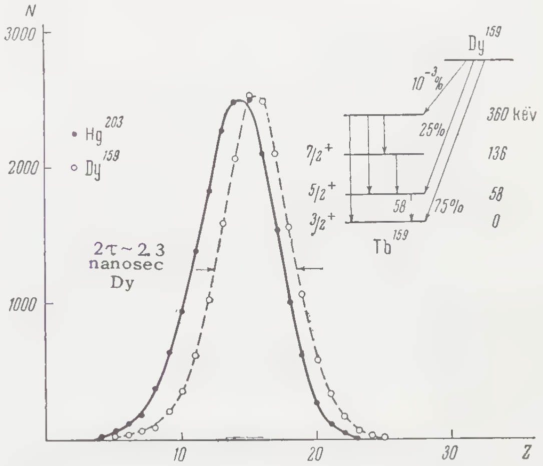

FIG. 1. Determination of the lifetime of the 58 kev level in Tb^{159}. N is the number of coincidences; Z is the channel number in the time analyzer. The channel width is 1.92×10^{-10} sec.

Having as a goal the complete removal of instrumental shifts in coincidence time (which become very significant in the measurement of such short times), we applied the method of "control of single pulses," which has been described in detail previously.[2]

The experimental values for the half-lives are:

$T_{1/2} = (1.3 \pm 0.4) \cdot 10^{-10}$ sec for the 58-kev level of Tb^{159},
$T_{1/2} = (3.8 \pm 0.5) \cdot 10^{-11}$ sec for the 78.7-kev level of Yb^{173}.

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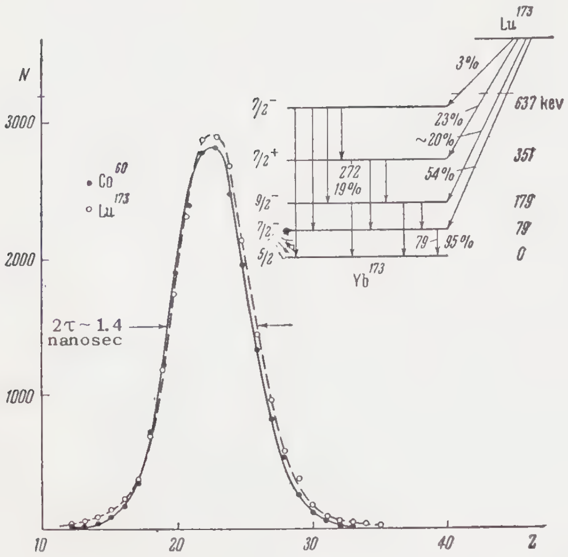

FIG. 2. Determination of the lifetime of the 78.7-kev level in Yb^{173}. The channel width is 2.16×10^{-10} sec.

The values for the g-factors are:

$$g_R = 0{,}44 \pm 0{,}10, \quad g_K = 1{,}37 \pm 0{,}08 \quad \text{for} \quad Tb^{159},$$
$$g_R = 0{,}35 \pm 0{,}04, \quad g_K = -0{,}52 \pm 0{,}02 \quad \text{for} \quad Yb^{173}.$$

The magnetic moments of Tb^{159} and Yb^{173} have been taken equal to 1.5 nuclear magnetons[4] and − 0.67 nuclear magnetons[5] respectively. The sign of the ratio $(g_K - g_R)/Q_0$ is known from work on angular correlation;[6,7] this permitted the choice of one of two possible sets of values for g_K and g_R.

In the case of Tb^{159}, our values for the g-factors differ markedly from the results of investigations by Coulomb excitation[8,6] (however, the results of the two works referred to also differ significantly from each other). We think that all the disagreement arises predominantly from the inaccurate knowledge of the E2 admixture, an account of which is important in the calculation of the g-factors from the results of experiments by Coulomb excitation.

In the case of Yb^{173} the g-factors obtained agree satisfactorily with the results of de Boer et al.[7] by Coulomb excitation. With the amount of admixture (4.5%) of the E2 component (this was well determined in the work of Romanov et al.[9]), the magnitude of the internal quadrupole moment of the nucleus Yb^{173} can be determined. $Q_0 = 8.4 \times 10^{-24}$ cm², in agreement with the results of other work,[8,10] and also with the magnitudes of quadrupole moments in this region of nuclei.

Finally, the values for the g-factors for collective motion which we determined are close to the value $g_R = Z/A$ which corresponds to the unified model value for a homogeneous charge distribution. The latter values are 0.41 for Tb^{159} and ~ 0.4 for Yb^{173}. However, the Coulomb excitation results for Tb^{159} give for this quantity 0.1 (reference 8) and 0.25 (reference 6), and 0.31 for (reference 7) Yb^{173}.

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GALVANOMAGNETIC PROPERTIES OF LITHIUM CHROMITE FERRITE

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The temperature dependences of the longitudinal and transverse galvanomagnetic effects in polycrystalline lithium chromite ferrite are determined. This ferrite exhibits a compensation point of the magnetic sublattices. Anomalies observed in the behavior of these effects above and below the compensation point are explained on the assumption that each magnetic sublattice of a ferrite makes a separate "characteristic" contribution to its galvanomagnetic properties.

1. The investigation of physical phenomena in ferrites having a compensation point Θ_c (the temperature at which the magnetic moments of the sublattices are "in balance"[1]) is important for the theory of ferrimagnetism. The individual characters of the different sublattices are especially pronounced near Θ_c, and it is easiest in this temperature range to determine the parts played by the different sublattices in producing ferrimagnetism and the accompanying effects (magnetoelastic, galvanomagnetic etc.) in ferrites.

We recently[2] reported the unusual behavior of the longitudinal galvanomagnetic effect in the ferrite $Li_2O \cdot 2.5\, Fe_2O_3 \cdot 2.5\, Cr_2O_3$, where this effect is reversed from negative to positive near Θ_c. Our attempted explanation was based on the assumption that each of the magnetic sublattices of the ferrite makes its own characteristic contribution to galvanomagnetic effects.[2]

The present paper reports detailed experimental data on even-order (longitudinal and transverse) galvanomagnetic effects in lithium chromite ferrite, with an interpretation based on the aforementioned hypothesis.

2. We prepared a ceramic ferrite with composition $Li_2O \cdot 2.7\, Fe_2O_3 \cdot 2.3\, Cr_2O_3$, which, like that mentioned in reference 2, had a compensation point. The sample was pre-annealed in air for four hours at 1000° C. The final sintering in air for three hours at 1200° C was followed by slow cooling. Electrical contacts were attached to the 5-mm cube with silver paste at 500° C. The galvanomagnetic effect $r = \Delta R/R$ was measured in a solenoid up to 2000 oe and in an electromagnet up to 12 000 oe, by comparison with a standard resistance in an unbalanced bridge. Since the resistance of the ferrite was strongly dependent on current strength, all measurements of r were obtained at 1 ma.

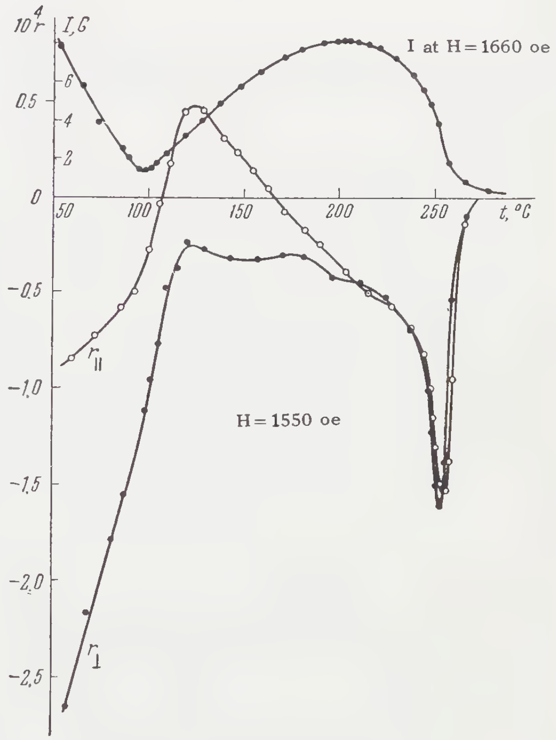

FIG. 1. Temperature dependence of magnetization I, and longitudinal ($r_\parallel$) and transverse ($r_\perp$) galvanomagnetic effects in the ferrite $Li_2O \cdot 2.7Fe_2O_3 \cdot 2.3Cr_2O_3$.

The magnetization I was measured ballistically in the solenoid. The temperature dependences of r and I from room temperature to 300° C were measured in an electric furnace with a bifilar winding of nonmagnetic nichrome. Since the galvanomagnetic properties of ferrites are strongly temperature dependent, the temperature was kept constant by placing the furnace in a molybdenum glass tube connected to a fore pump. The furnace current was stabilized electronically. The temperature of the sample was measured with a

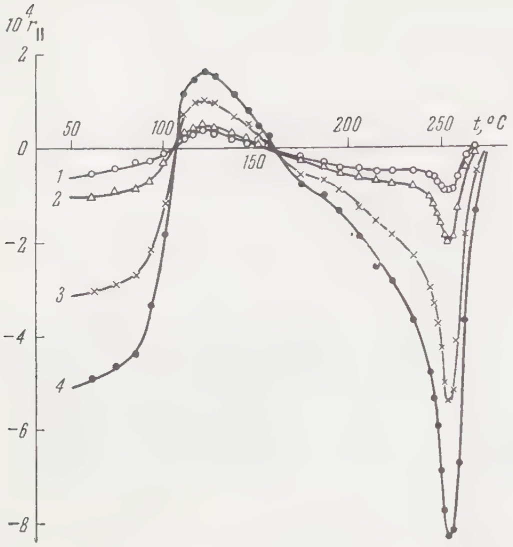

FIG. 2. Temperature dependence of the longitudinal galvanomagnetic effect in $Li_2O \cdot 2.7Fe_2O_3 \cdot 2.3Cr_2O_3$ in different magnetic fields: 1 – 1080 oe, 2 – 2260 oe, 3 – 6800 oe, 4 – 11 070 oe.

copper-constantan thermocouple and a PPTV-1 high-resistance potentiometer.

3. The galvanomagnetic effects and magnetization were measured at first in relatively small fields (in the solenoid). Figure 1 shows the temperature dependences of the longitudinal ($r_{\parallel}$) and transverse ($r_{\perp}$) galvanomagnetic effects (at H = 1550 oe) and magnetization I (at H = 1660 oe) for a sample of $Li_2O \cdot 2.7\, Fe_2O_3 \cdot 2.3\, Cr_2O_3$.

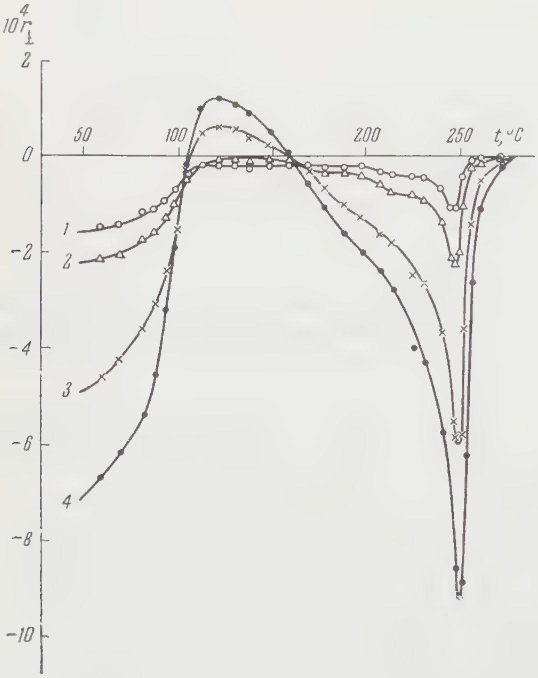

FIG. 3. Temperature dependence of the transverse galvanomagnetic effect in $Li_2O \cdot 2.7Fe_2O_3 \cdot 2.3Cr_2O_3$ in different magnetic fields: 1 – 1080 oe, 2 – 2260 oe, 3 – 6800 oe, 4 – 11 070 oe.

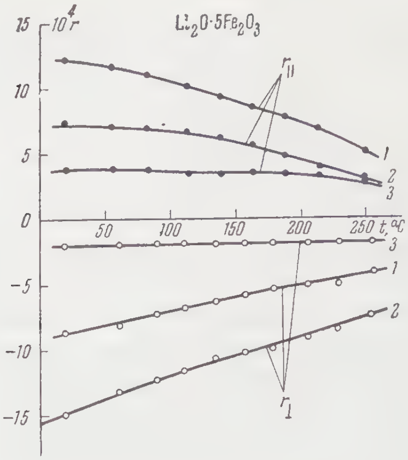

FIG. 4. Temperature dependences of $r_{\parallel}$ and $r_{\perp}$ in the ferrite $Li_2O \cdot 5Fe_2O_3$: 1 – 2080 oe, 2 – 10 370 oe, 3 – 6000 oe.

The magnetization curve shows that although complete compensation of the magnetic sublattices does not occur, the compensation point Θ_c is easily found at 98° C. The temperature dependences of $r_{\parallel}$ and $r_{\perp}$ show that below Θ_c the galvanomagnetic effects have the same sign in both longitudinal and transverse fields; this is similar to observations on magnetite[3] and many ferrites. Above Θ_c the signs of the effects differ; this corresponds to the "normal" behavior characteristic of most ferromagnetic materials.

In addition to the identical (negative) signs of $r_{\parallel}$ and $r_{\perp}$, their anomalous behavior below Θ_c lies in the fact that the transverse effect is almost three times as large as the longitudinal effect. Above Θ_c, $r_{\parallel}$ is larger in absolute value than $r_{\perp}$; this is the normal behavior. In a certain temperature range the so-called second rule of even-order effects, $r_{\perp} = -\frac{1}{2}\, r_{\parallel}$, is observed.

As the Curie point is approached a galvanomagnetic effect r_p due to spontaneous magnetization (the "para-process"), the magnitude and (negative) sign of which are independent of the applied

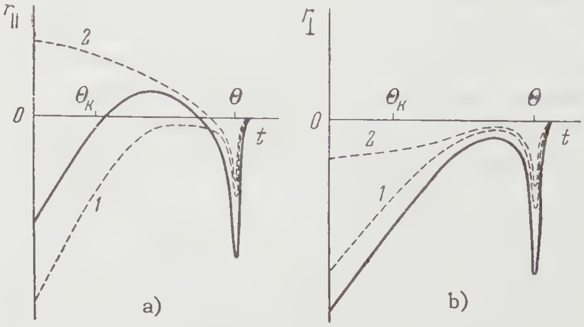

FIG. 5. Schematic temperature variation of the galvanomagnetic effects in lithium chromite ferrites; a – longitudinal, b – transverse galvanomagnetic effect.

field direction, is superposed on the $r_{\parallel}$ and $r_{\perp}$ effects due to technical magnetization. The negative maximum of r_p is located precisely at the Curie point, where $r_{\parallel} \rightarrow 0$ and $r_{\perp} \rightarrow 0$.

Figures 2 and 3 show the temperature dependences of $r_{\parallel}$ and $r_{\perp}$ in stronger fields up to about 12 000 oe. The behavior shown in Fig. 1 is repeated on the whole. Below Θ_c, $r_{\perp} > r_{\parallel}$, and both effects are negative. At 112° C ($\Theta_c = 98°$ C) the longitudinal effect becomes positive in all fields. The transverse effect is negative in not very strong fields (as in Fig. 1), but becomes positive in stronger fields. We are at present making a detailed investigation of this behavior, which is not yet understood. It should also be noted that at the Curie point the negative maximum of the "para-process" galvanomagnetic effect is of smaller magnitude in Fig. 2 than in Fig. 3.

4. The galvanomagnetic anomalies can be accounted for by individually different characteristic galvanomagnetic properties of the sublattices in a given ferrite. These differences are due to the nature and arrangement of the sublattice ions. The galvanomagnetic effects in the sublattices are superposed to produce the galvanomagnetic effect of the ferrite as a whole, with a temperature dependence that, as we have seen, exhibits very complicated and anomalous behavior in the case of chromite ferrite.

According to Gorter,[1] lithium chromite ferrite contains two sublattices, one with Fe, Cr, and some Li ions in octahedral sites, and the other with Fe and Li ions in tetrahedral sites. Below Θ_c the octahedral sublattice dominates the magnetic behavior of the ferrite; its galvanomagnetic properties are, according to our hypothesis, anomalous and "magnetite-like" ($r_{\parallel}$ and $r_{\perp}$ having like signs). Above Θ_c the tetrahedral sublattice dominates; it has "normal" galvanomagnetic properties, $r_{\parallel}$ being positive and $r_{\perp}$ being negative. It is interesting that our measurements (Fig. 4) show the galvanomagnetic properties of the simple ferrite $Li_2O \cdot 5Fe_2O_3$ to be "normal," i.e., above Θ_c lithium chromite ferrite has the same galvanomagnetic properties as $Li_2O \cdot 5Fe_2O_3$. The dashed schematic curves in Fig. 5 depict the temperature behavior of the $r_{\parallel}$ and $r_{\perp}$ effects in the octahedral and tetrahedral sublattices; the solid curves represent the combined effect for the ferrite. The scheme of Fig. 5 furnishes a very good explanation of our experimental findings given in Figs. 1 – 3. It appears that 1) when $T < \Theta_c$ the transverse effect $r_{\perp}$ is much greater than $r_{\parallel}$ and both are negative; 2) when $T > \Theta_c$, $r_{\parallel}$ is reversed from negative to positive, the zero point of $r_{\parallel}$ (the compensation point Θ'_c for the longitudinal galvanomagnetic effect) does not necessarily coincide with Θ_c, and the position of Θ'_c is determined by the behavior of the dashed curves around Θ_c in Fig. 5; 3) at the Curie point the "para-process" galvanomagnetic effect has a somewhat greater negative maximum in Fig. 5b than in Fig. 5a. This difference is accounted for by the fact that $r_{\parallel}$ and $r_{\perp}$ do not disappear completely at the Curie point. In Fig. 5a the $r_{\parallel}$ effect due to technical magnetization processes (the positive effect) partially cancels the maximum of r_p, whereas in Fig. 5b, since $r_{\perp}$ and $r_{\parallel}$ are both negative, the maximum negative effect is somewhat greater at the Curie point.

Our experimental investigation of the galvanomagnetic properties of lithium chromite ferrite furnishes additional evidence that the ferrite contains magnetic sublattices with different temperature dependences of spontaneous magnetization.

The disclosure and study of the galvanomagnetic properties of each magnetic sublattice will help to reveal the character of the electrical properties of ferrite crystals.

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THE DENSITY OF H_2-D_2 *MIXTURES*

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The density of the binary system H_2-D_2 has been measured in the range from the melting point to 20.4°K, and the relative excess volumes $\Delta V/V$ determined. For all mixtures $\Delta V/V < 0$ and is independent of temperature within the experimental accuracy. For an equimolar mixture $\Delta V/V \approx -0.01$. The theoretical calculations of $\Delta V/V$ for the H_2-D_2 system[9,11] do not agree with the experimental data.

INTRODUCTION

THE influence of quantum effects on the microscopic properties of matter can be revealed by a study of the physical properties of mixtures of isotopes. As is well known, isotopic mixtures behave ideally in any phase according to classical considerations,[1] so that a contribution from quantum effects to the thermodynamic functions is characterized by a departure from ideality. These effects are most marked for isotopes of the light elements at low temperatures. The properties of the He^3-He^4 system have been studied fairly thoroughly,[2] but for H_2-D_2 mixtures only the liquid-vapor[3] and liquid-solid[4] diagrams are known. The data obtained indicate a considerable departure in the behavior of these systems from the laws for ideal mixtures. This is shown especially clearly in the investigations of Kogan, Lazarev and Bulatova[4] who found a separation of a mixture of H_2 and D_2 into two phases at temperatures below the melting point.

The present work was undertaken for the further study of the deviation of the hydrogen isotopes system from ideality. The density of mixtures of H_2 and D_2 was measured in order to calculate the excess mixing volume

$$\Delta V = V - (c_H V_H + c_D V_D); \qquad (1)$$

where V, V_H and V_D are the molar volumes of the mixture, of hydrogen and of deuterium and c_H and c_D are the molar concentrations of hydrogen and deuterium.

EXPERIMENTAL SECTION

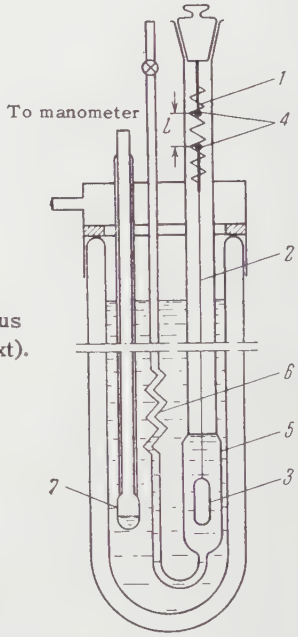

FIG. 1. Diagram of the apparatus (the description is given in the text).

The measurements were made by the method of hydrostatic weighing on a spring balance (Fig. 1). A quartz float 3 of weight ≈200 mg and volume ≈0.05 cm³ is hung from the spring 1 by a 0.03 mm diameter copper wire. The distance L between the quartz markers 4 attached to the lower and upper ends of the spring was measured with a KM-6 cathetometer. If L is the distance between the markers when the float is immersed in liquid, and L_0 the corresponding distance in vacuum, then $L_0 - L = k\rho$, where ρ is the liquid density and k is a coefficient of proportionality, depending on the sensitivity of the spring and the volume of the float.

The spring was made of 0.08 mm diameter tungsten wire and had 130 turns of diameter ~ 8 mm, which gave a sensitivity of about 50 cm/g. After the spring had been prepared it was kept under a load of 200 mg to prevent any zero drift. During the measurements this drift did not exceed 0.01 mm per day. The pure isotopes H_2 and D_2

T°, K	ΔV/V, %	T, °K	ΔV/V, %	T, °K	ΔV/V, %	T, °K	ΔV/V, %
c_D = 91.1%		c_D = 76.95%		c_D = 61.6%		c_D = 57.3%	
20.39	0.52	20.46	1.14	20.46	1.11	20.40	0.95
20.09	0.52	19.82	1.08	19.96	1.00	19.89	1.00
18.36	0.53	19.79	1.02	19.20	1.05	19.79	0.93
		19.08	0.90	19.15	1.10	19.10	1.04
		18.58	0.96	18.44	1.06	19.08	0.99
		18.44	1.03	18.67	0.95	19.09	0.98
		18.39	0.98	17.52	1.05	18.51	0.98
		17.79	0.96	16.97	1.09	18.38	0.98
		17.75	0.96			17.82	0.91
c_D = 47.75%		c_D = 38.3%		c_D = 20.35%		17.61	0.98
20.44	1.22	20.43	1.0	20.38	0.86	16.78	0.83
19.81	1.16	19.69	1.12	19.83	0.94	16.76	0.89
19.80	1.27	19.67	1.05	19.76	0.81	16.74	0.82
19.26	1.15	19.03	1.00	19.23	0.91	c_D = 9.7%	
19.24	1.14	18.99	1.00	19.00	0.82	20.42	0.70
18.46	1.19	18.38	1.06	18.55	0.89	14.97	0.71
18.42	1.10	18.31	1.09	18.29	0.78	14.86	0.74
17.79	1.16	17.71	1.07	17.73	0.90	14.48	0.85
17.70	1.18	17.52	1.14	17.58	0.85		
17.11	1.15	16.85	0.99	16.81	0.72		
17.06	1.13	16.77	0.97	16.64	0.78		
16.38	1.19	16.10	1.04	16.05	0.70		
16.36	1.14	15.84	1.08	15.46	0.85		
				15.13	0.75		

were used to determine the magnitude of k. The stretch of the spring was measured with vessel 5 filled with liquid hydrogen or deuterium at various temperatures, and from the known densities we found

$$k = 25.09 \pm 0.02 + (0.040 \pm 0.004)(L_0 - L)\ \mathrm{cm}^4/\mathrm{g}.$$

This method of calibration removes additional errors related to the determination of the float volume, and reduces the error arising from the action of surface tension on the support 2. We note that according to the data of Forziati et al.[5] the latter effect is less than the experimental errors. A dependence of k on $L_0 - L$ is explained by the change in sensitivity of the spring with elongation, since the angle between the coils increases for large elongations. Several repeat calibrations between the measurements of the mixtures gave the same values of k, within the limits of error.

The measurement of the elongation introduced a noticeable error in addition to the calibration errors. The errors associated with maintaining and measuring the temperature and with the preparation of the mixtures were negligibly small. The total error in measuring the density was 0.1 — 0.2%.

In preparing the mixtures, the hydrogen and deuterium were thoroughly cleaned by being passed through a glass coil cooled in liquid hydrogen. Further purification took place on condensing the mixture into the apparatus through the spiral 6. Mixing was achieved by many successive condensations and evaporations. The concentrations of the mixtures were calculated from the partial pressures. The temperature was obtained by pumping liquid hydrogen, and was measured by the saturated vapor pressure of pure hydrogen, contained in vessel 7.

The measurements were made in the range from the melting point of the mixture to 20.4° K, both for the temperature decreasing by steps and increasing. The hydrogen and deuterium had the room temperature equilibrium ortho-para concentrations. No noticeable change in density due to the ortho-para conversion was found, even after keeping for several hours. This was evidently due to the high purity of the hydrogen and the absence of any catalysts. To achieve these conditions, the investigated mixture was replaced by a fresh one when the smallest trace of cloudiness appeared in coil 6.

The good agreement between results obtained with a given mixture on different days indicates the absence of appreciable HD formation in the mixtures.

The densities of eight mixtures were measured with D_2 concentrations from 10 to 90%. From the densities obtained, the molar volumes of the mixtures $V = \mu/\rho$ (μ is the molecular weight of a mixture) were calculated, and the excess mixing

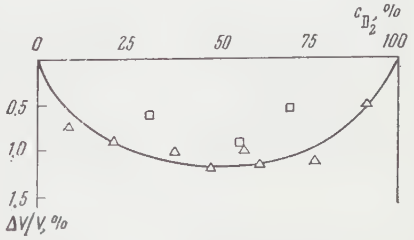

FIG. 2. The dependence of the excess mixing volume on the concentration of the mixture at T = 20.4°K: Δ – data of the present work, □ – Lambert's data.[6]

volume derived from (1). In the calculation account was taken of the difference between the concentration of the liquid and that of the original mixture, due to the presence of the vapor. The correction introduced into the concentrations did not exceed 0.15%. The values of V_D for temperatures below the melting point of D_2 were calculated from an extrapolation formula.

The values of $\Delta V/V$ for all the mixtures studied are shown in Table I. The concentrations given in the table are mean values. The true concentrations do not differ by more than 0.05% from the means, due to various corrections to the gas phase. Figure 2 shows the dependence of $\Delta V/V$ on concentration at a temperature $T = 20.4°K$. It can be seen from the table that the relative excess volume is independent of temperature within the limits of error, and the $\Delta V/V(c)$ isotherms for other temperatures have the same appearance as shown in Fig. 2. The figure also shows data taken from a preliminary report by Lambert,[6] who measured ΔV at 20.3°K. The results agree qualitatively, but the difference is apparently somewhat greater than the errors in measuring (the accuracy of the measurements is not stated by Lambert).

DISCUSSION OF THE RESULTS

The experimental results obtained indicate a relatively large departure from ideality for mixtures of the hydrogen isotopes. Over the whole range of temperature and concentration studied the mixing volume is negative, and is of the same order of magnitude as for binary systems of liquefied gases[7] such as O_2-Ar, O_2-N_2 etc. Simon and Bellemans[9] have recently published a theoretical calculation of $\Delta V/V$ for isotopic mixtures, based on the work of Prigogine et al.[8] They obtain $\Delta V/V \approx 1\%$ for an equimolar mixture of H_2 and D_2, i.e., the calculated volume is positive, which is contrary to the experimental data. We should note that the calculations for the He^3-He^4 and D_2-H_2 systems according to Progogine's theory agree qualitatively with the experimental data.[8,10]

Blagoĭ, who studied mixtures of liquefied gases, found good agreement between the experimental data and calculations made according to the formula proposed by Meares.[11] Meares considered that the total volume change on mixing two liquids of approximately equal molar volumes consists of two terms

$$\Delta V = \Delta V_{\rm I} + \Delta V_{\rm II}. \qquad (2)$$

The first term, which is related to the energy of mixing, was calculated by Scatchard.[12] According to Scatchard

$$\Delta V_{\rm I} = \beta \Delta E_v, \qquad \Delta E_v \approx \Delta H - T\alpha\Delta V/\beta, \qquad (3)$$

where β is the compressibility of the mixture, ΔE_V is the excess mixing energy at constant volume, ΔH is the excess mixing enthalpy and α is the thermal expansion coefficient of the mixture. The second term, which arises because of the difference in compressibilities and internal pressures, p, was obtained by Biron[13] and was somewhat refined by Meares:[11]

$$\Delta V_{\rm II} = \gamma c_1 c_2 (\beta_1 V_1 - \beta_2 V_2)(p_1 - p_2), \qquad (4)$$

where γ is a constant coefficient for all liquids. By comparing the experimental and theoretical data, Meares found $\gamma = 0.4$.

In view of the smallness of ΔV, we can put $\Delta E_V \approx \Delta H$ to a first approximation in calculating the mixing volume. According to Lambert's data,[6] $\Delta H = 12\, c_H c_D$ cal/mole for hydrogen and deuterium. The values of α, β and V for the pure isotopes were taken from the compilation of Woolley et al.,[14] and were found from the additivity rule for the mixtures. The internal pressure was determined from the thermodynamic relation $p = T\alpha/\beta$. For an equimolar mixture at $T = 20.3°K$ we found $\Delta V_I = 0.11$ cm^3/mole, $\Delta V_{II} = -0.15$ cm^3/mole, so that $\Delta V = -0.04$ cm^3/mole. This value agrees as to sign with experiment, but is about five times smaller in absolute magnitude. It is interesting to note that the same result can be obtained if ΔV is calculated according to (3) by substituting the value of ΔE_V, derived from Hildebrandt's formula for regular mixtures:

$$\Delta E_v = (c_H V_H c_D V_D / V)\,[(Q_H/V_H)^{1/2} - (Q_D/V_D)^{1/2}]^2,$$

where Q_H and Q_D are the heats of vaporization of hydrogen and deuterium.

Better agreement with experiment can be obtained if the value of the coefficient in (4) is taken as close to unity instead of 0.4, although it is unlikely that such a choice can be justified.

We can thus see that the existing theory of mixtures cannot explain the appreciable compression found experimentally on the formation of mixtures of the isotopes H_2 and D_2.

In conclusion, we may remark that the phase separation of the hydrogen isotopes near the melting point can be explained by considering the correlation in the solid state.[15] It is possible that similar effects in the liquid must be taken into account for a correct calculation of the volume change on mixing.

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TRANSITION FROM AN ANTIFERROMAGNETIC STATE TO A WEAKLY FERROMAGNETIC STATE IN A MAGNETIC FIELD

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The dependence of the magnetic moment of anhydrous cobalt sulfate on temperature and field strength is studied in detail for different crystallographic directions. For the c axis the temperature dependence has been determined for the critical field in which the transition occurs from the antiferromagnetic to the ferromagnetic state. In order to account for the anomalous magnetic properties observed in $CoSO_4$ and previously in $CuSO_4$, the thermodynamic potential is analyzed for the D_{2h}^{16} symmetry group to which the investigated compounds belong. The rapid increase of the susceptibility anisotropy near T_N for $T > T_N$ and the anomalously large value of $\chi_\perp$ below T_N are explained qualitatively.

1. INTRODUCTION

INVESTIGATIONS of the magnetic properties of polycrystalline anhydrous sulfates of Ni^{++}, Co^{++}, Fe^{++}, and Cu^{++}[1,2] have shown that all these compounds are antiferromagnetic. The temperature dependence of the susceptibility of $CoSO_4$ and $CuSO_4$ (Fig. 1 in reference 1 and Fig. 3 in reference 2) exhibits an anomalous increase near the respective transition points in the paramagnetic region.

Single crystals of the same compounds were used in magnetic studies for the purpose of investigating the anomaly in detail. The results obtained for $CuSO_4$, which have been reported in reference 3, show that the sharp rise of the susceptibility above the transition point T_N and its anomalously large value below T_N are observed only for $\chi_\perp$, while $\chi_\parallel$ exhibits ordinary behavior. Preliminary results for a $CoSO_4$ single crystal have also been published,[4] showing that the transition to the antiferromagnetic state is anomalous only along the c axis. At T = 4.2°K anomalous dependence of the magnetic moment on the applied field is observed along the c axis; this is associated with a transition from the antiferromagnetic to the ferromagnetic state. The fact that the maximum spontaneous magnetic moment in $CoSO_4$ is ~30% of the nominal moment calculated assuming complete freezing of orbital moments, and the small value of the critical field required for the transition to ferromagnetism, exclude the possibility that the observed transition in a field can be attributed to the complete destruction of antiferromagnetic ordering.

The present paper reports a more detailed experimental investigation of the anomaly in $CoSO_4$ from 1.3 to 15°K. It is also shown that the anomalies for $CoSO_4$ and $CuSO_4$ above T_N are associated with the fact that in anhydrous sulfates antiferromagnetic ordering combined with weak ferromagnetism can be established.

2. $CoSO_4$ SAMPLES AND MEASUREMENTS

The magnetic properties of $CoSO_4$ single crystals were investigated. The samples were prepared by evaporating a solution of cobalt sulfate in molten ammonium sulfate.*[3] The growth process of $CoSO_4$ crystals is more reliable than that of $CuSO_4$ crystals; the process requires 8 to 10 hours at 430° C. The $CoSO_4$ crystals are dark violet and are very well faceted, so that they can be oriented by face reflections. They are also less hygroscopic than $CuSO_4$. However, in time a transition to a different crystal modification occurs accompanied by color change from dark violet to pink and by crumbling to a powder. The magnetic properties of the powder differ from those of the crystals, but we have not studied the former in detail.

Our largest $CoSO_4$ samples weighed 1.0 to 1.5 mg, with linear dimensions up to 1 mm. These crystals were rectangular bipyramids. In most

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*The author is greatly indebted to N. N. Mikhaĭlov for his assistance in preparing $CoSO_4$ single crystals.

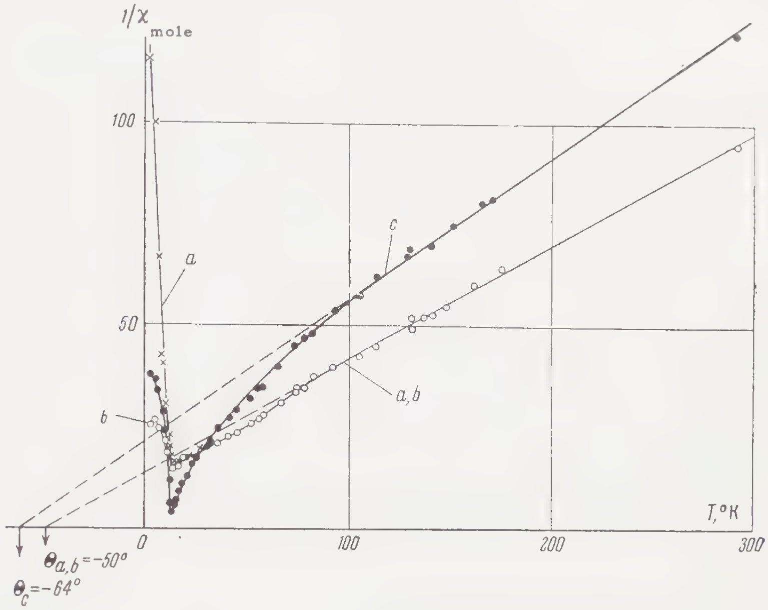

FIG. 1. Temperature dependence ot the reciprocal molar susceptibility of $CoSO_4$ along the a, b, and c axes.

instances the two pyramids were shifted relative to each other along the a axis. The structure of anhydrous cobalt sulfate has not yet been investigated thoroughly. Hammel's[5] x-ray study of powdered samples revealed an orthorhombic lattice with four molecules in the unit cell. The lattice constants are a = 8.46 A, b = 6.66 A, and c = 4.65 A.* Recent x-ray studies of $CuSO_4$ and $ZnSO_4$ single crystals[6] showed that these compounds both belong to the D_{2h}^{16} space group. We thus have a basis for the hypothesis that $CoSO_4$ belongs to the same symmetry group.

The triaxial character of our $CoSO_4$ crystals was confirmed goniometrically. The b axis passes through the vertex of the pyramid perpendicular to its base; the a and c axes are parallel to the edges of the base. At the vertex opposite faces form angles of 65° and 70°. These angles are in best agreement with the assumption that the faces are (210) and (011) planes. The c axis is then parallel to the faces forming the 65° angle, and the a axis is perpendicular to the c axis.

Our magnetic measurements were obtained with the apparatus described in reference 7. The range of magnetic field strengths was broadened by the use of pole pieces such as those employed by Sucksmith.[8] The magnetic field at the second maximum of $H\,\partial H/\partial z$ was thus increased to 18 koe. However, since $H\,\partial H/\partial z$ in this region is considerably smaller than at the upper maximum, the sensitivity of the apparatus was somewhat reduced. Small moments were thus measured less accurately. As a result, at high temperatures (above ~ 60°K), where the magnetic moments are small, the accuracy of relative measurements was 5 to 6%; at low temperatures it was 2%.

We studied a few $CoSO_4$ single crystals weighing from 0.5 to 1.5 mg. A crystal with proper orientation (attained by x-ray or goniometric means) was attached to the quartz rod of the suspension with BF cement. The accuracy of orientation and suspension was 2 to 3°. Magnetic susceptibility was measured along all three crystal axes at temperatures from 1.3 to 300°K.

At all temperatures the susceptibility is field-independent up to ~ 4 koe. Figure 1 shows the temperature dependence of the reciprocal molar susceptibility of $CoSO_4$. From 300° down to 14 – 18°K the susceptibilities along the a and b axes coincide. In almost this entire temperature range $\chi_{a,b} > \chi_c$; the sign of the susceptibility anisotropy is reversed only at $T \approx 27°K$. The Curie-Weiss law is obeyed for both directions in the approximate range 100 – 300°K. Along the c axis we have the law $\chi = 2.89/(T+64)$ with the g-factor 2.48; along the b axis we have $\chi = 3.59/(T+50)$ with the g-factor 2.77.

Figure 1 in reference 4, which gives the temperature dependence of the molar susceptibility of $CoSO_4$ from 1.3 to 70°K, shows that the maximum susceptibility along all three axes is attained at $T_N = 12°K$. The anisotropy increases sharply and there is a very sharp susceptibility peak along the c axis at this temperature. Below 12° the susceptibility differs along the three axes. It is interesting that at $T \rightarrow 0°K$ the susceptibility does not

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*In the present paper the notation for the $CoSO_4$ lattice constants differs from that in reference 1 and 5, so that the orientation of $CoSO_4$ can correspond to that of $CuSO_4$ given in reference 6.

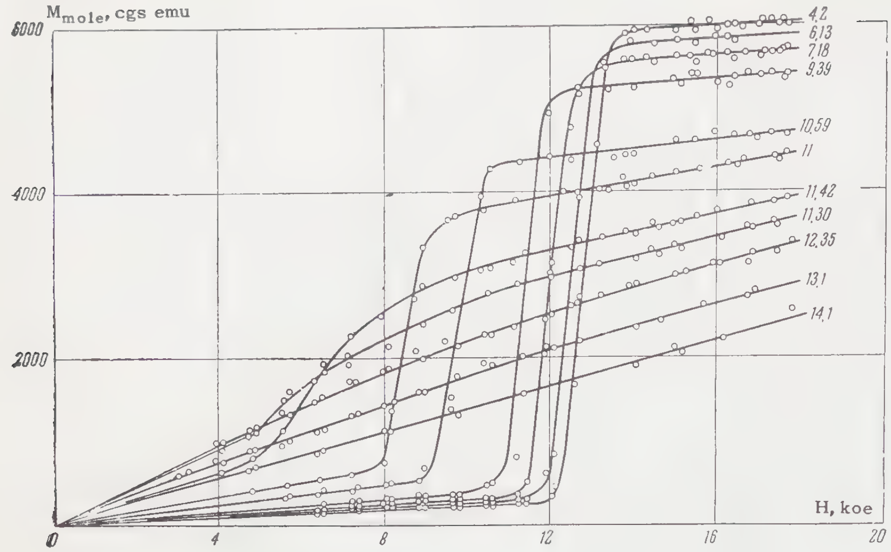

FIG. 2. Magnetic moment of $CoSO_4$ along the c axis vs magnetic field at different temperatures (indicated in °K at the end of each curve).

approach exactly zero along any one of the axes.

The results obtained in large fields parallel to the c axis are of greatest interest. Although susceptibility along the a and b axes is field-independent up to 18 koe, magnetic properties along the c axis exhibit a sharp anomaly in the antiferromagnetic region. Figure 2 shows the field dependence of the molar magnetic moment along this axis at different temperatures. Up to a value H_{cr} the moment increases linearly with the field. This is followed by a sharp increase of the moment up to a value M_s, after which it increases slowly with H. The M(H) curves and M_s and H_{cr} depend on temperature; as the temperature increases the transition region is broadened, while H_{cr} and M_s are diminished.

Slight nonlinearity in the field dependence of the magnetic moment is also observed above the antiferromagnetic transition point and disappears completely only at 14°K. The sharpest field-dependent increase of the moment occurs at the lowest temperatures; below 4°K the M(H) curves coincide within error limits. The moment increases by a factor of almost 20 (from 300 to 5900 cgs emu) as the field changes from 12 to 13.2 koe. We obtained a maximum moment of 6000 cgs emu.

The value of the maximum moment diminishes with time; in our samples the change amounted to ~10% after two weeks. This reduction is apparently associated with a partial transition to a different crystal modification.

The temperature variation of the magnetic moment is shown in Fig. 3. The peak of the M(T) curve is seen to shift toward lower temperatures as the field strength increases. The peak is shifted by 6°K with a field increase of ~9 koe. The M(T) curves have no peak in fields $\gtrsim 14$ koe and are saturated at low temperatures.

Figure 4 shows how the force acting on a sample depends on the magnetic field direction in different fields at $T = 4.2$°K. Curve 1 pertains to the field below H_{cr}, curve 2 to the transition region, and curve 3 to the maximum field. Curve 3 exhibits flat-topped peaks.

3. THEORY

Anomalous increase of the magnetic moment as the field is strengthened, similar to that observed for $CoSO_4$, had been observed earlier by Shalyt[9] in polycrystalline $FeCl_2$, and by Bizette et al.[10] in a single crystal. However, $FeCl_2$ is a layer-type antiferromagnet with a strong ferromagnetic intralayer interaction and weak antiferromagnetic interaction between layers; this is revealed by the ferromagnetic sign of the constant Θ in the Curie-Weiss law. The ferromagnetic moment is here associated with flipping of the moments in layers,[11] which is entirely natural for the given structure even in weak fields. Similar magnetic behavior has been observed in the intermetallic compound $MnAu_2$[12] and in the rare-earth elements Dy, Ho, and Er.[13-15]

$CoSO_4$ is the first ionic crystal having an antiferromagnetic sign of Θ where destruction of the initial antiferromagnetic structure was observed in a relatively weak field ($\mu H \ll kT$). We shall show that this effect and the anomaly observed earlier in the temperature variation of $CuSO_4$ susceptibility[3] are associated with the fact that antiferromagnetic ordering combined with weak ferromagnetism can be established in anhydrous sulfates. We shall apply the theory of second-

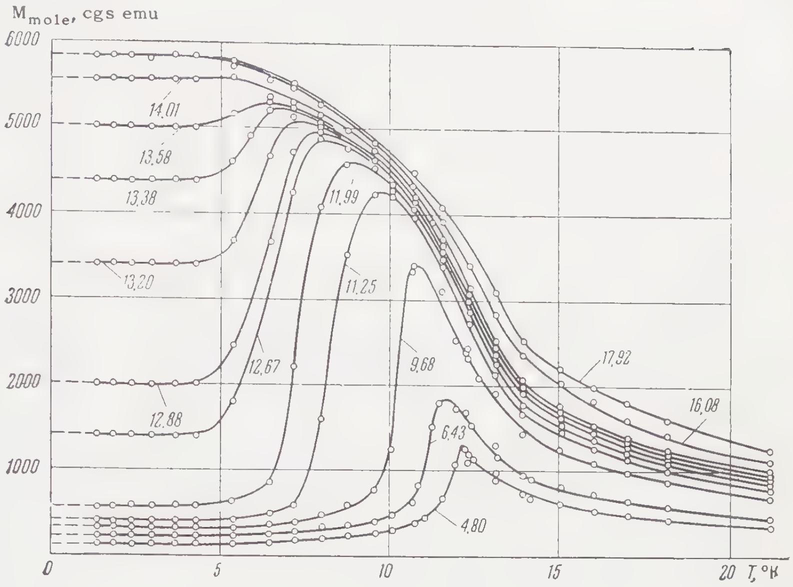

FIG. 3. Magnetic moment of $CoSO_4$ along the c axis vs temperature in different fields (indicated in kilo-oersteds along the curves).

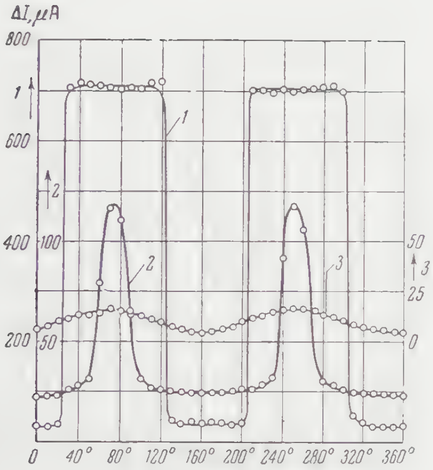

FIG. 4. Force acting on a sample vs angle between the magnetic field and c axis, in different fields. 1 – H = 17.9 koe; 2 – H = 12.7 koe; 3 – H = 8 koe.

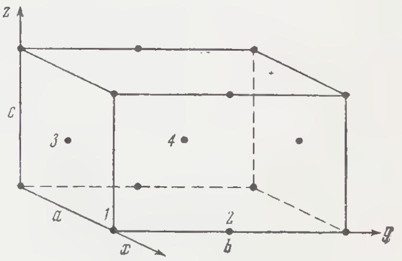

FIG. 5. Metallic ions in sulfate unit cell. For $CuSO_4$ a = 8.39 A, b = 6.69 A, c = 4.83 A; for $CoSO_4$ a = 8.46 A, b = 6.66 A, and c = 4.65 A.

order phase transitions to this problem, as has been done by Dzyaloshinskii.[16]

$CuSO_4$ crystals possess the symmetry of the D_{2h}^{16} space group, with four metallic ions in each unit cell arranged as shown in Fig. 5. We introduce the vectors $\mathbf{m}$, $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$, defined by

$$\mathbf{m} = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 + \mathbf{s}_4,$$

$$\mathbf{l}_1 = \mathbf{s}_1 - \mathbf{s}_2 - \mathbf{s}_3 + \mathbf{s}_4, \quad \mathbf{l}_2 = \mathbf{s}_1 - \mathbf{s}_2 + \mathbf{s}_3 - \mathbf{s}_4,$$

$$\mathbf{l}_3 = \mathbf{s}_1 + \mathbf{s}_2 - \mathbf{s}_3 - \mathbf{s}_4. \tag{1}$$

Here $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$ are the respective spins of the four ions. $\mathbf{m}$ is obviously the mean magnetic moment of the unit cell, and $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ are antiferromagnetic vectors corresponding to the different possible magnetic structures.

When the representation of the D_{2h}^{16} group formed by the twenty vectors in (1) is transformed into irreducible representations we find that this symmetry group permits nine types of antiferromagnetic ordering, three of which are purely antiferromagnetic (A), while the other six possess weak ferromagnetism (AWF). The types of ordering are listed in the table.

When the irreducible representations transforming the vectors in (1) are known, the most general expansion of the thermodynamic potential

Types of antiferromagnetic ordering* permitted by the D_{2h}^{16} space group

State	l_1	l_2	l_3	m	State	l_1	l_2	l_3	m
A_1	l_{1x}				AWF_{1y}	l_{1z}			m_y
A_2		l_{2z}			AWF_{2y}		l_{2x}		m_y
A_3			l_{3y}		AWF_{1z}	l_{1y}			m_z
AWF_{2x}		l_{2y}		m_x	AWF_{3z}			l_{3x}	m_z
AWF_{3x}			l_{3z}	m_x					

*The type of order is indicated by the subscripts 1, 2, 3; the spin direction is indicated by x, y, z.

to second-order terms is easily obtained, as follows:

$$\begin{aligned}\Phi = {} & \tfrac{1}{2} A_1 \mathbf{l}_1^2 + \tfrac{1}{2} A_2 \mathbf{l}_2^2 + \tfrac{1}{2} A_3 \mathbf{l}_3^2 + \tfrac{1}{2} Bm^2 + \tfrac{1}{2} a_1' l_{1x}^2 \\ & + \tfrac{1}{2} a_1'' l_{1y}^2 + \tfrac{1}{2} a_2' l_{2x}^2 + \tfrac{1}{2} a_2'' l_{2y}^2 + \tfrac{1}{2} a_3' l_{3x}^2 + \tfrac{1}{2} a_3'' l_{3y}^2 \\ & + \tfrac{1}{2} b_1 m_x^2 + \tfrac{1}{2} b_2 m_y^2 + \beta_1 m_x l_{2y} + \beta_2 m_x l_{3z} + \beta_1' m_y l_{1z} \\ & + \beta_2' m_y l_{2x} + \beta_1'' m_z l_{1y} + \beta_2''' m_z l_{3x} + \gamma_1 l_{2y} l_{3z} \\ & + \gamma_2 l_{1z} l_{2x} + \gamma_3 l_{1y} l_{3x} + \gamma_4 l_{1x} l_{22} + \gamma_5 l_{2z} l_{3y} + \gamma_6 l_{1x} l_{3y}. \end{aligned} \tag{2}$$

The last six terms in this expansion of Φ show that none of the states listed above is realized in its pure form, but each one is always accompanied by some admixture. We shall hereafter assume this effect to be negligibly small, without affecting our results qualitatively.

The foregoing types of ordering for the D_{2h}^{16} group were first obtained by Bozorth,[17] who used them to account for the ferromagnetism of orthoferrites. Turov and Naĭsh[18] made a detailed study of the magnetic properties of orthoferrites in external fields, based on an analysis of (2). However, unlike these investigators, we shall assume that in the absence of a magnetic field one of the pure antiferromagnetic states (for definiteness, A_2) is realized below the transition point. We shall be interested in determining whether the application of a magnetic field induces other states possessing weak ferromagnetism.

A field perpendicular to and a field parallel to the direction of antiferromagnetic ordering present two qualitatively different cases. In the first case ($\mathbf{H} \perp Oz$) the field induces the AWF_{2x} and AWF_{2y} states with weak ferromagnetism but without reordering of the spins. We can therefore put $\mathbf{l}_1 = \mathbf{l}_3 = 0$ and write the potential in the considerably simpler form

$$\begin{aligned}\tilde{\Phi} = {} & \tfrac{1}{2} A\mathbf{l}^2 + \tfrac{1}{2} B\mathbf{m}^2 + \tfrac{1}{2} a_1 l_x^2 + \tfrac{1}{2} a_2 l_y^2 + \tfrac{1}{2} b_1 m_x^2 + \tfrac{1}{2} b_2 m_y^2 \\ & + \beta_1 m_x l_y + \beta_2 m_y l_x + \tfrac{1}{4} Cl^4 - \mathbf{mH}. \end{aligned} \tag{3}$$

Here the most important fourth-order exchange term has been taken into account.

Using the scheme proposed by Borovik-Romanov and Ozhogin,[19] we obtain the following solutions for **m** and **l**:

$$m_x = \frac{H_x}{B + b_1} - \frac{\beta_1 l_y}{B + b_1}, \quad m_y = \frac{H_y}{B + b_2} - \frac{\beta_2 l_x}{B + b_2}, \quad m_z = \frac{H_z}{B};$$

$$l_x = \frac{\beta_2 H_y}{\beta_2^2 - (B + b_2)(A + a_1 + Cl^2)},$$

$$l_y = \frac{\beta_1 H_x}{\beta_1^2 - (B + b_1)(A + a_1 + Cl^2)},$$

$$l_z (A + Cl^2) = 0, \qquad l^2 = l_x^2 + l_y^2 + l_z^2. \tag{4}$$

The coefficient A = 0 at the transition point T_N; near T_N we shall, as usual, assume

$$A = \lambda (T - T_N). \tag{5}$$

It is also useful to introduce two characteristic temperatures:

$$T_1 = T_N - \lambda^{-1} (a_1 - \beta_2^2/(B + b_2)),$$
$$T_2 = T_N - \lambda^{-1} (a_2 - \beta_1^2/(B + b_1)), \tag{6}$$

which determine the ordering energy in the states AWF_{2y} and AWF_{2x}, respectively. It follows from our assumptions regarding the magnitudes and signs of the constants in (3) that $T_N > T_1$ and $T_N > T_2$.

Above the transition point ($T > T_N$) $l_z \equiv 0$. Neglecting Cl^4 and using (6), the equations in (4) finally become

$$m_x = \left[\frac{1}{B + b_1} + \frac{\beta_1^2}{(B + b_1)^2 (T - T_2) \lambda}\right] H_x,$$

$$m_y = \left[\frac{1}{B + b_2} + \frac{\beta_2^2}{(B + b_2)^2 (T - T_1) \lambda}\right] H_y, \quad m_z = \frac{H_z}{B};$$

$$l_x = \frac{\beta_2 H_y}{(B + b_2) \lambda (T_1 - T)}, \quad l_y = \frac{\beta_1 H_x}{(B + b_1) \lambda (T_2 - T)}, \quad l_z = 0. \tag{7}$$

Below the transition point ($T < T_N$) $l_z \neq 0$; consequently, in (4), $A + Cl^2 = 0$ and $l^2 = -A/C$. We then obtain

$$m_x = \frac{H_x}{B + b_1 - \beta_1^2 / a_2}, \quad m_y = \frac{H_y}{B + b_2 - \beta_2^2 / a_1}, \quad m_z = \frac{H_z}{B};$$

$$l_x = \frac{\beta_2 H_y}{\beta_2^2 - (B + b_2) a_1}, \quad l_y = \frac{\beta_1 H_x}{\beta_1^2 - (B + b_1) a_2}, \quad l_z^2 = l^2 - (l_y^2 + l_x^2). \tag{8}$$

The equations in (7) show that in the given case a magnetic field also induces antiferromagnetic ordering above T_N. It can be seen from (7) that the magnetic moment varies linearly with the field, and that the paramagnetic susceptibility increases sharply as the transition point is approached. Below the transition point the susceptibility is constant and greater in absolute magnitude than in the absence of weak ferromagnetism.

It is shown by (8) that below the transition point an external magnetic field rotates the vector **l** (directed along the z axis when H = 0) toward the xy plane; l_x and l_y are then proportional to H_y and H_x, respectively. In a certain field H_{cr}, **l** lies entirely in the xy plane and l_z vanishes. In the special cases $H = H_x$ and $H = H_y$ the critical fields are

$$H_{cr} = (\beta_1^2 - (B + b_1) a_2) l / \beta_1,$$
$$H_{cr} = (\beta_2^2 - (B + b_2) a_1) l / \beta_2. \tag{9}$$

When l_z vanishes the condition $A + Cl^2 = 0$ [in (4)] is not required, and l can vary with the field. We obtain the following dependence of l_x on H_y near the transition point:

$$l_x = \sqrt{\lambda (T_1 - T)/C} + \beta_2 H_y/2 (B + b_2) \lambda (T_1 - T). \qquad (10)$$

Substituting (10) in the expression for m_y in (4), we obtain a relationship that is typical for antiferromagnets possessing weak ferromagnetism:

$$m_y = \sigma + \chi H. \qquad (11)$$

At low temperatures in strong fields l becomes large and the expansion (3) cannot be used. However, by expanding in powers of the unit vector γ in the direction of l, we can derive equations similar to (8), (9), and (11).

By comparing the expressions for m_y in (11) and (8) we learn the following. l_x in a weak field H_y is accompanied by a supplementary ferromagnetic moment along the y axis. In fields smaller than H_{ycr} this has the sole effect of increasing the magnetic susceptibility along that axis compared with ordinary paramagnetic susceptibility. In fields greater than H_{ycr}, m_y contains a field-independent term representing a spontaneous ferromagnetic moment. Thus when the field is perpendicular to the direction of antiferromagnetic ordering a state with weak ferromagnetism can arise. The mechanism responsible for this state is associated with the rotation of the antiferromagnetic vector into the plane perpendicular to its original direction.

In the second case, when the field is parallel to the direction of antiferromagnetic ordering, the table shows that simple rotation of the antiferromagnetic vector l_2 cannot produce weak ferromagnetism in this direction. A change of structure and a transition to the state AWF_{1z} $(l_{1y} \neq 0)$ or AWF_{3z} $(l_{3x} \neq 0)$ are required. The energy of one of these states must differ very little (~1°K) from that of the original state A_2. We shall assume that this is true for the state AWF_{1z}.

In accordance with the foregoing assumptions and using the notation $l_{1y} = l_1$ and $l_{2z} = l_2$, the potential (2) can be expressed by

$$\widetilde{\Phi} = \tfrac{1}{2}(A_1 + a)\, l_1^2 + \tfrac{1}{2} A_2 l_2^2 + \tfrac{1}{2} B m_z^2 - \beta m_z l_1 + \tfrac{1}{4} C_1 l_1^4$$
$$+ \tfrac{1}{4} C_2 l_2^4 + \tfrac{1}{2} D l_1^2 l_2^2 - m_z H_z. \qquad (12)$$

The principal fourth-order exchange terms have here been taken into account. By minimizing $\widetilde{\Phi}$ we obtain two solutions:

$$l_2 = 0, \qquad \frac{\beta}{l_1 B} H_z = C_1 l_1^2 + \lambda_1 (T - T_1), \qquad m_z = \frac{H_z + \beta l_1}{B};$$
$$l_2^2 = -[\lambda(T - T_N) + D l_1^2]\, C_2^{-1}, \qquad (13a)$$

$$\frac{\beta}{l_1 B} H_z = C_1 l_1^2 \left(1 - \frac{D^2}{C_1 C_2}\right) + \lambda_1 (T - T_1) - \frac{D}{C_2} \lambda (T - T_N),$$
$$m_z = \frac{H_z + \beta l_1}{B}, \qquad (13b)$$

where

$$\lambda (T - T_N) = A_2, \qquad \lambda_1 (T - T_1) = A_1 + a - \beta^2/B. \qquad (14)$$

Equation (13a) shows that the result for the paramagnetic region is entirely analogous to that obtained for perpendicular fields. We therefore have

$$m_z = [1/B + \beta^2/B\, (T - T_1)\lambda_1]\, H_x. \qquad (15)$$

The complexity of the foregoing equations necessitated a graphical method of solution for $T < T_N$. With empirically determined coefficients in Eqs. (13a) and (13b) and making certain assumptions concerning the region in which solutions exist, we obtained graphically $l_1(H)$ at different temperatures and $l_1(T)$ in different fields (Figs. 6 and 7), as well as the region in which the state A_2 $(l_2 \neq 0)$ exists for different fields and temperatures. Figure 7 shows that below T = 11°K there exists for each temperature a critical field in which the value of l_1 jumps discontinuously. l_2 vanishes at the same point, i.e., the critical field transforms the material from an ordered mixture of A_2 $(l_2 \neq 0)$ and AWF_{1z} $(l_1 \neq 0)$ to a pure AWF_{1z} $(l_1 \neq 0)$ state possessing weak ferromagnetism.

At temperatures from T_N (11.7°) to T_1 (11°) in fields from 0 to 4.5 koe the state A_2 is reached by a second-order transition; the $l_1(H)$ and $l_1(T)$ curves exhibit kinks but no jumps. Below T_1 in fields above 4.5 koe, A_2 is reached by a first-order transition; the $l_1(H)$ and $l_1(T)$ curves exhibit jumps in these regions. It should be noted that in this case $\chi_{||}$ will not approach zero when

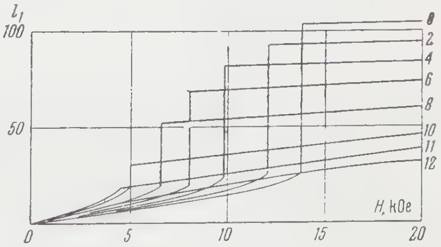

FIG. 6. Field dependence of the antiferromagnetic vector $\mathbf{l}_1$ at different temperatures (indicated in °K at the end of each curve).

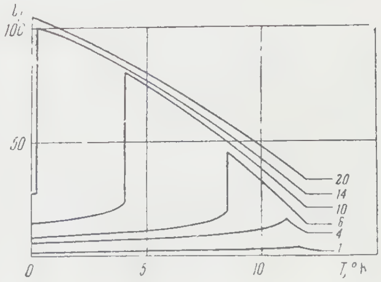

FIG. 7. Temperature dependence of the antiferromagnetic vector $\mathbf{l}_1$ in different fields (indicated in kilo-oersteds at the end of each curve).

$T \rightarrow 0°K$ even in weak fields. This follows readily from the potential (12) without fourth-order terms at low temperatures and $\mathbf{l}_1^2 + \mathbf{l}_2^2 + \mathbf{m}^2 = \text{const}$.

For the case $\mathbf{H} \parallel Oz$ we were forced to make a number of assumptions regarding the magnitudes of the expansion coefficients. The results therefore represent only one of the possible special cases in the behavior of the given crystals. Reordering of the structure is also possible in the case $\mathbf{H} \perp Oz$. Since the expansion (12) is valid only around T_N, the theoretical curves in Figs. 6 and 7 can be regarded as only very rough extrapolations far from the transition point.

Our theoretical results lead to the following conclusion. A field can induce weak ferromagnetism even in a crystal that is purely antiferromagnetic in the absence of the field. This can occur only if the symmetry group of the crystal permits weak ferromagnetism in zero field in addition to the pure antiferromagnetic state. In this case increasing susceptibility will always be observed along the direction permitting weak ferromagnetism as T_N is approached from higher temperatures.

The foregoing formulas hold true around but not too near T_N, according to the limitations introduced by Borovik-Romanov and Ozhogin.[19] The anomalies predicted by the formulas will be observed only for sufficiently large values of the constant β, which is responsible for weak ferromagnetism. Closeness of T_i to T_N is another requirement.

4. DISCUSSION OF RESULTS

The results obtained for the antiferromagnetic compounds $CoSO_4$ and $CuSO_4$ will now be discussed and compared with the theory.

1. Near the transition point where $T > T_N$ the susceptibility increases sharply along the c axis for $CoSO_4$ (Fig. 1 in reference 4) and along the a and b axes for $CuSO_4$ (Fig. 3 in reference 3). $\chi_a = \chi_b$ for the latter. The preceding theoretical discussion shows that this behavior can be accounted for by field-induced antiferromagnetic ordering near T_N. The temperature dependence of the susceptibility for this case is given by (7) or (15). For the purpose of comparing the experimental results with these equations we have plotted the temperature variation of $1/\Delta\chi$ for $CuSO_4$ and $CoSO_4$ (Fig. 8). For the former compound $\Delta\chi$ represents the difference $\chi_\perp - \chi_\parallel$, for the latter it represents $\chi_c - \chi^*$, where χ^* is an extrapolation of the Curie-Weiss law for χ_c. χ^* is somewhat arbitrary in the case of $CoSO_4$, since the Curie-Weiss law is ordinarily not observed by Co^{++} ions at low temperatures. However, the resulting error will not be very large since, first, $\Delta\chi \gg \chi^*$, and, secondly, χ^* should not change appreciably in the given small temperature interval.

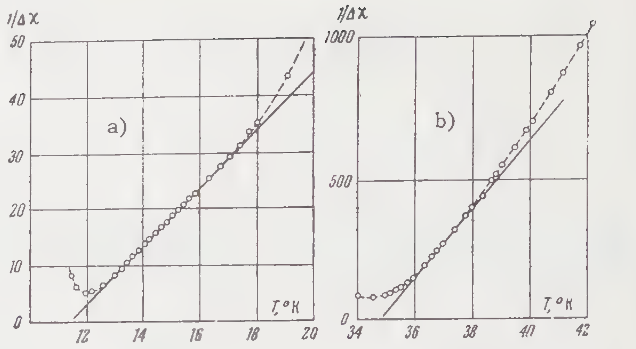

FIG. 8. Temperature dependence of $1/\Delta\chi$ near T_N. a – for $CoSO_4$; b – for $CuSO_4$.

The linear portions of the curves in Fig. 8 indicate the qualitative agreement of the experimental results with theoretical formulas over a 2° interval for $CuSO_4$ and a 4° interval for $CoSO_4$. This makes it possible to determine the temperature T_1 associated with the energy of the corresponding weakly ferromagnetic states that induce the anomalous rise of χ. Taking $T_N = 12°$ for $CoSO_4$ and 35.5° for $CuSO_4$, we obtain $T_N - T_1 = 0.6°$ for $CoSO_4$ and $T_N - T_1 = 0.8°K$ for $CuSO_4$. The experimental curves of $\chi_\parallel$ and $\chi_\perp$ for $CuSO_4$ exhibit peaks at different temperatures. It was reasonable to take the temperature for the maximum of $\chi_\parallel$ as T_N, since according to (10) the maximum of $\chi_\perp$ must depend strongly on the applied field, as has actually been observed experimentally.[2,3]

The results obtained at temperatures above T_N thus show that, although both substances perform a transition to pure antiferromagnetism at T_N, in each of them one of the states with weak ferromagnetism differs very little energetically (< 1°K) from pure antiferromagnetism. The behavior for $T < T_N$ must be considered in determining the actual states.

2. In the case of $CuSO_4$ (Fig. 3 in reference 3) the temperature dependence of χ_c below T_N indicates that the c axis is the axis of easy magnetization, and that the transition to the state A_2 therefore occurs when $H = 0$. The anomalous behavior of $\chi_\perp$ in $CuSO_4$ can therefore be accounted for by a mechanism for the generation of weak ferromagnetism, associated with rotation of the antiferromagnetic vector in a field. In these fields below H_{cr}, $\chi_\perp$ should have an unusually large value [see Eq. (8)]; this was observed experimentally.

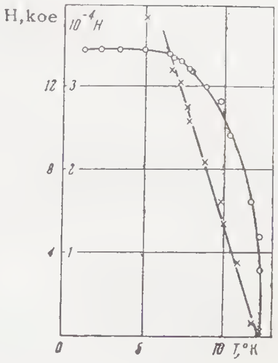

FIG. 9. Temperature dependence of H_{cr}. ○ – the function $H_{cr}(T)$ (left-hand scale); × – the function $H_{cr}^4(T)$ (right-hand scale).

The experimental data for $CuSO_4$ enable us to estimate some expansion coefficients of the thermodynamic potential, as well as the critical field, from the formula $\frac{1}{2}\chi H_{cr}^2 = k(T_N - T_1)$. We obtain $B = 270$, $\beta^2/a = 188$, and $H_{cr} \approx 100$ koe.

The observed behavior of the susceptibility of $CuSO_4$ is similar to that of CoF_2.[20,21] Dzyaloshinskii[22] has shown that the symmetry group of CoF_2 allows a weakly ferromagnetic state. We can therefore assume that CoF_2 is also subject to a mechanism producing weak ferromagnetism, associated with the rotation of the antiferromagnetic vector in a field.

3. In the case of $CoSO_4$ we do not arrive at such a definite conclusion regarding the direction of spontaneous magnetization. The two most essential facts for $T < T_N$ are: a) χ does not approach zero along any axis when $T \to 0$, and b) the magnetic moment along the c axis increases sharply at a certain strength of the external field. From a comparison with item 2) we conclude that $CoSO_4$ is in state A_2 at $H = 0$. One of the weakly ferromagnetic states AWF_{1z} or AWF_{3z} should therefore result when a magnetic field is applied along a magnetization direction of the sublattices.

The available experimental data do not enable us to determine what state actually exists. The foregoing conclusions regarding the states of $CoSO_4$ are confirmed by the good qualitative agreement between experiment (Figs. 2 and 3) and theory (Figs. 6 and 7). It should be noted that, since l_1 and m_z are related as in (13), the field and temperature dependences of m_z are represented by the curves in Figs. 6 and 7. It must be remembered that the theoretical curves are only very rough extrapolations at a large distance from the transition point.

Figure 9 gives the temperature dependence of H_{cr}, which for definiteness was taken to be the field at the upper knee of the M(H) curves in Fig. 2. According to the foregoing discussion these curves depict the transition from the mixture of states A_2 and AWF_{1z} to the state AWF_{1z}. $m_{max}H_{cr}$ must obviously be of the order of the energy difference between A_2 and AWF_{1z}. $T_N - T_1 = 0.8°K$ obtained in this manner agrees well with the temperature dependence of χ in the paramagnetic region. This result is $\sim 7\%$ of the total exchange energy; as pointed out in our introduction, this excludes the possibility of accounting for the observed transition in a field as resulting from complete destruction of antiferromagnetic ordering. Figure 9 shows that the temperature dependence of H_{cr} over a broad temperature range is represented by the empirical law

$$(H_{cr}/H_{0cr})^4 = 1 - \alpha T. \tag{16}$$

Two conclusions result from the rotation diagrams in Fig. 4. First, the ferromagnetic moment disappears at angles where the projection of the external field on the c axis equals H_{cr}. Secondly, the broad flat peaks of the curves apparently indicate that the ferromagnetic moment can form an appreciable angle with the c axis.

The results for $CoSO_4$ show that although the crystal structure of $CoSO_4$ is unknown, the theoretical results derived from a consideration of space group D_{2h}^{16} are in good agreement with experimental findings for this substance.

In conclusion the author wishes to thank A. S. Borovik-Romanov for his continued interest and guidance. The author is also grateful to Academician P. L. Kapitza for his continued interest, to I. E. Dzyaloshinskii for several valuable suggestions and helpful discussions, and to V. I. Kolokol'nikov for assistance with the measurements.

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NUCLEAR MAGNETIC RESONANCE IN ELASTICALLY DEFORMED ROCK SALT

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The effect of elastic deformation on the nuclear magnetic resonance absorption line of Na^{23} was studied in rock salt crystals. Components of the tensor relating the elastic deformation of the lattice to the gradient of the nuclear electric field were determined and found to be $S_{11} = \pm 2 \times 10^{15}$ and $S_{44} = \mp 0.5_5 \times 10^{15}$ cgs esu.

THE investigation of quadrupole effects in nuclear magnetic resonance (NMR) makes it possible to obtain an amount of information on defects in solids. The defects create additional gradients of the electric field in the crystalline lattice, and the interaction of the nuclei with these gradients leads to a change in the NMR absorption lines. Excitation of additional gradients is brought about on the one hand by lattice distortions, i.e., by local changes in the symmetry of the lattice close to the defect, and on the other by the electric fields, which the defects create in those cases where they possess a charge.

In order for it to be possible to estimate the amount of lattice distortion near the defects by the results of NMR studies, a knowledge of the connection between the elastic deformations of the lattice and the nuclear electric field gradient is obviously necessary. This connection can be found from NMR experiments in elastically deformed crystals, and has recently been determined by Shulman et al.[1] for In^{115} in InSb.

Let us consider what is the effect of elastic deformation on the NMR absorption line. Elastic deformation changes the symmetry of the crystal; consequently, the electric field gradient in the lattice also changes. In a cubic crystal, such as NaCl, the field gradient is equal to zero by virtue of the high symmetry; in this case, the NMR line, consisting of the central line and satellites, is observed as a single "total" absorption line.

Elastic deformation, lowering the symmetry of the cubic crystal, leads to the generation of a field gradient which produces a splitting of the absorption line into its components. In principle, for sufficiently large elastic deformations in the cubic crystal, complete splitting of the components is possible. In reality, however, the gradients obtained in practically attainable elastic deformations are very small and lead not to a splitting of the components of the line, but only to its broadening.

Shulman et al.,[1] introduce the tensor C, which relates the nuclear electric field gradient with the elastic stresses in the lattice, and investigate its properties. It is of great interest to consider the tensor S which relates the tensor of the nuclear field gradient φ with the elastic deformations in the lattice δ. This tensor can be introduced by analogy with the tensor C in the following way:

$$\varphi_{\mu\nu} = \sum_{\varkappa,\lambda} S_{\mu\nu,\varkappa\lambda}\,\delta_{\varkappa\lambda} \qquad (\varkappa,\ \lambda,\ \mu,\ \nu = x,\ y,\ z) \tag{1}$$

For cubic crystals, there are only two independent components of the tensor S, which in the universally adopted notation are written as S_{11} and S_{44}. These components, as will be shown below, can be determined by measurement of the broadening of the NMR absorption line in an elastically deformed crystal. As objects of investigation in the present work, we used specimens of rock salt with dimensions of approximately 8×8×20 mm, cut along the [001], [110] and [111] directions.

The specimens were placed in the coil of an NMR oscillator circuit[2] and were elastically deformed by linear compression along the specified direction; the axis of compression in this case was perpendicular to the direction of the magnetic field. The maximum loading on the specimens amounted to 66 kg/cm^2. For each specimen, the derived absorption lines of Na^{23} were recorded for different loadings and angles of rotation of the specimen around the compression axis relative to the magnetic field.

The effect of elastic deformation on the absorption line was weak, as expected. In elastically deformed specimens, only a reversible decrease in the maximum of the derived absorption line was observed, without a noteworthy change in its width, within the limits of experimental error. The largest decrease amounted to about 20 percent.

For the determination of the components of the tensor S, it is necessary to connect this decrease with the frequency shift of the satellites. If we assume that the elastic deformation produces only a shift of the satellites without change in their shape, then, by making use of the property of additivity of the second moment of the line, we can write

$$\Delta\nu^2 = \Delta\nu_0^2 + 0.6\Delta\nu_c^2,$$

where $\Delta\nu_0^2$ and $\Delta\nu^2$ are the second moments of the absorption line in the undeformed and deformed specimens, respectively; $\Delta\nu_c$ is the frequency shift of the satellites, and the coefficient 0.6 takes into account the relative intensity of the satellites.

Analysis of the readings showed that the shape of the absorption line, both in the initial and in the deformed specimens, is very close to Gaussian. For the Gaussian shape $\delta\nu^2 = 4\Delta\nu^2$, where $\delta\nu$ is the breadth of the line measured between the points of maximum and minimum of the derivative. Taking this into account, we have

$$\delta\nu^2 = \delta\nu_0^2 + 2.4\Delta\nu_c^2.$$

The intensity of the absorption line is proportional to $A\delta\nu^2$, where A is the maximum of the derivative, and the coefficient of proportionality is determined by the shape of the line.

Inasmuch as the intensity of the absorption line should not change in elastic deformation, and its shape remains Gaussian, as has already been noted, $A\delta\nu^2 = A_0\delta\nu_0^2$, and A and A_0 are the maxima of the derivatives of the absorption line in the deformed and undeformed crystals, respectively.

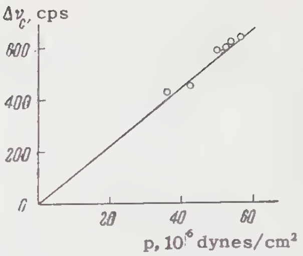
By use of the latter relation, we obtain the final formula for the frequency shift of the satellites:

$$\Delta\nu_c = \pm\delta\nu_0\sqrt{(A_0/A-1)/2.4}. \quad (2)$$

The dependence of the frequency shift of the satellites $\Delta\nu_c$, calculated by this formula, is shown in Fig. 1, from a loading for a specimen with the compression axis along the [001] direction. As expected, the frequency shift is proportional to the loading.

Figures 2 and 3 show the dependence of $\Delta\nu_c$ on the angle of rotation of the specimen α for a constant load. In the first case (Fig. 2) the compression axis is directed along [111], while the angle α is the angle between the direction of the magnetic field and the [1$\bar{1}$0] direction. In the second case, (Fig. 3), the compression axis is directed along [110], while the angle α is the angle between the direction of the magnetic field and the [001] direction. All the points in Figs. 1 — 3 represent the mean of 4 or 5 independent measurements.

FIG. 1. Loading dependence of the frequency shift of the satellite for a specimen with the compression axis along the [001] direction.



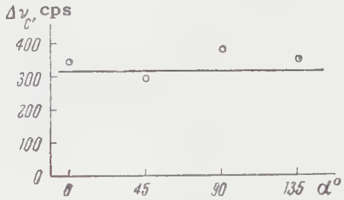


FIG. 2. Dependence of the frequency shift of the satellites on the angle between the direction of the magnetic field and the [1$\bar{1}$0] direction for a specimen with compression axis along [111]. Loading is 65 × 10^6 dynes/cm².

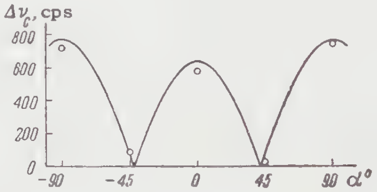


FIG. 3. Dependence of the frequency shift of the satellites on the angle between the direction of the magnetic field and the [001] direction for a specimen with the compression axis along [110]. Loading is 61 × 10^6 dynes/cm².

As is seen from the drawings, for a deformation along [111] there is no angular dependence (the same holds for a specimen with the compression axis along [001]), while for a deformation along [110], a clear cut angular dependence is observed.

The formula for the frequency shift of the satellites can also be obtained by means of perturbation theory:[3]

$$\Delta\nu_c = 3eQ\,(2m-1)\,\varphi_{HH}/4I\,(2I-1)\,h,$$

where I — spin of the nucleus, Q — its quadrupole moment, m — magnetic quantum number, φ_{HH} — component of the nuclear gradient along the direction of the magnetic field. For Na^{23}, $I = 3/2$ and the formula for the frequency shift of the satellites has the form

$$\Delta\nu_c = \pm(eQ/2h)\,\varphi_{HH}. \quad (3)$$

Use of Eqs. (2) and (3) makes it possible to obtain the components of the tensor of the gradient φ_{HH} in an elastically deformed crystal from the experimental results. Further, expressing φ_{HH}

in terms of the tensor S and the elastic strain tensor δ, we can determine the components of S in accord with (1).

It should be noted that inasmuch as for the cubic crystals only two independent components of the tensor S exist, it suffices for their determination to measure the effect of elastic deformation only for a specimen with the axis of compression along [110] for two different values of α. Nevertheless, with the aim of checking the results, we thought it desirable to carry out measurements for specimens with the compression axis along three different directions.

Let us consider how the component of the tensor gradient φ_{HH} can be expressed for each case separately. It must be kept in mind that the tensor φ is a symmetric tensor of second rank and can be put into diagonal form if its principal axes are known.

1. Deformation along [001]. For such a deformation, the cubic crystal becomes tetragonal, and the principal axes of the tensor gradient are the cube axes from considerations of symmetry.

If the x, y, z axes are directed along [100], [010], and [001], then the tensor gradient is diagonal and its components are equal to

$$\varphi_{xx} = \varphi_{yy} = -\tfrac{1}{2}\varphi_{zz} = \tfrac{1}{2} S_{11}(s_{12} - s_{11})P,$$

where P is the stress and the elastic constants of the crystal are denoted by s.

The direction of the magnetic field lies in the xy plane and the component of the tensor gradient along this direction is equal to

$$\varphi_{HH} = \varphi_{xx}\sin^2\alpha + \varphi_{yy}\cos^2\alpha = \varphi_{xx} = \varphi_{yy}$$

(α — angle between the direction of the magnetic field and the [010] direction).

Thus φ_{HH}, and consequently the frequency shift of the satellites, in accord with (3), does not depend on α, which, as has been noted above, is the situation actually observed in experiment.

2. Deformation along [111]. The cubic crystal becomes trigonal. The tensor of the gradient along the principal axes xyz, which are $[11\bar{2}]$, $[1\bar{1}0]$ and [111], is diagonal and its components are:

$$\varphi_{xx} = \varphi_{yy} = -\tfrac{1}{2}\varphi_{zz} = -\tfrac{1}{3} S_{44}s_{44}P.$$

In this case $\varphi_{HH} = \varphi_{xx} = \varphi_{yy}$ also do not depend on α — the angle between the direction of the magnetic field and the $[1\bar{1}0]$ direction. As is seen from Fig. 2, this is confirmed by experiment.

3. Deformation along [110]. The cubic crystal becomes orthorhombic with the three axes of second order $[1\bar{1}0]$, [001], and [110]. The same axes are the principal axes of the tensor gradient. Denoting them by x, y, z, we have

$$\varphi_{xx} = \left[\tfrac{1}{4} S_{11}(s_{11} - s_{12}) - \tfrac{1}{2} S_{44}s_{44}\right]P,$$

$$\varphi_{yy} = -\tfrac{1}{2} S_{11}(s_{11} - s_{12})P,$$

$$\varphi_{zz} = \left[\tfrac{1}{4} S_{11}(s_{11} - s_{12}) + \tfrac{1}{2} S_{44}s_{44}\right]P.$$

It is easy to see that in this case φ_{HH} changes as a function of α with a period of 180°:

$$\varphi_{HH} = -\tfrac{1}{2}\left[\varphi_{zz} + (\varphi_{zz} + 2\varphi_{xx})\cos 2\alpha\right],$$

where α is the angle between the direction of the magnetic field and the [001] direction. Such a character of the dependence of $\Delta\nu_c$ on α is actually observed in experiment (Fig. 3).

Thus, in all three cases, the experimental results are identical with the predictions of the theory. This supplies a basis for assuming that decrease in the maximum of the derivative of the absorption line which was observed by experiment is actually connected with the shift in the frequency of the satellites brought about by elastic deformation.

Upon substitution of the expressions obtained for φ_{HH} in Eq. (3), the best correspondence with experimental results is obtained for $S_{11} = \pm 2 \times 10^{15}$ and $S_{44} = \mp 0.5_5 \times 10^{15}$ in cgs esu units.

The curves in Figs. 1 — 3 represent the theoretical dependences calculated by means of Eq. (3) for given values of the components of the tensor S.

Unfortunately, in view of the smallness of the observed effects, the results for S_{11} and S_{44} are not sufficiently accurate. According to our estimates, the random errors in these values amount to ± 15 percent. Moreover, it is evident that small systematic errors take place connected with the fact that the shape of the absorption line is not exactly Gaussian and probably changes slightly under elastic deformation.

In conclusion, the author thanks M. I. Kornfel'd for discussion of the results, O. M. Nilov for help in the measurements and V. V. Sokolov for carrying out mechanical tasks.

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A BOSON DOUBLET EQUATION

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An equation for a boson doublet is set up. The equation is shown to be invariant to two transformations of the Pauli-Gürsey type.

THE question as to whether bosons are described by wave equations of the Klein-Gordon form or by Kemmer-Duffin equations has not yet received a conclusive answer. It is therefore interesting to investigate the physical content and mathematical properties of both types of equation both for individual particles and for multiplets. The corresponding equation for Fermion doublets has been analyzed in sufficient detail by Gürsey.[1] One of the most interesting properties of this equation is its invariance to the Pauli-Gürsey transformation group. In this work, we examine the question as to whether or not there exists a doublet boson equation which is invariant to transformations of the type of the Pauli-Gürsey group. A doublet boson equation could have immediate practical application, since the K mesons have a manifest doublet structure.

The simultaneous linearization of two Klein-Gordon equations gives the first-order equation

$$(B_\nu \partial/\partial x_\nu + k_0 I)\,\psi = 0, \tag{1}$$

with which we will describe the boson doublet. Here

$$B_\nu = 1\times\beta_\nu, \qquad I = 2\sigma_1\times 1, \tag{2}$$

where β_ν (and later η_ν) are 5×5 or 10×10 Kemmer-Duffin matrices[2] and σ_k are the 2×2 spin matrices. The sign × designates the direct product of matrices.

Equation (1) can be shown to be invariant under the transformations

$$\psi' = \exp\,(i\lambda + i\lambda_k T_k)\cdot\psi, \tag{3}$$

where λ and λ_k are four real parameters and

$$T_1 = \sigma_1\times 1, \qquad T_2 = \sigma_2\times\eta_5, \qquad T_3 = \sigma_3\times\eta_5;$$

$$\eta_5 = \eta_1\eta_2\eta_3\eta_4. \tag{4}$$

The matrices T_k ($k = 1, 2, 3$) can be regarded as isotopic spin matrices. The transformation (3) gives a conservation law for the number of bosons and conservation laws for the isotopic spin components.

The electric charge operator has the form

$$Q = e\,(T_3 + \tfrac{1}{2}). \tag{5}$$

The equation for the boson in an external electromagnetic field, A_ν, will be

$$\left[B_\nu\left(\frac{\partial}{\partial x_\nu} + \frac{i}{\hbar c}\,QA_\nu\right) + k_0 I\right]\psi = 0. \tag{6}$$

The charge-conjugate wave function is

$$\psi^C = C\bar{\psi}^T, \qquad \bar{\psi} = \psi^*\,(1\times\eta_4),$$

where the operator C can be one of the operators

$$C_1 = 2\sigma_3\times 1, \qquad C_2 = 1\times\eta_5. \tag{7}$$

Equation (1) is invariant under the canonical transformations

$$\psi' = a\psi + bIC_2\bar{\psi}^T, \qquad \psi' = a\psi + bC_2\bar{\psi}^T,$$

$$|a|^2 + |b|^2 = 1, \tag{8}$$

which are analogous to the Pauli transformation.

One possible assumption is that equation (6) describes the K meson doublet (K^+, K^0). If that is the case, it is not necessary to include K^- and $\bar{K}^0$ mesons explicitly in the scheme of elementary particles, since they can be obtained by charge conjugation. The place thus vacated in the scheme can be used for new mesons (for example, for D^- and D^0 mesons). Then the new mesons can be particles with strangeness −1, and not necessarily with strangeness +2, as in the Gell-Mann scheme.

We note further that the equation for the meson doublet can be written in a form which takes into account the mass difference of the mesons:[3]

$$[B_\nu\,\partial/\partial x_\nu + k_0 I \exp\,(fT_3)]\,\psi = 0. \tag{9}$$

Here f is a real constant. Equation (9) is invariant under the transformation

$$\psi' = \exp\,(i\lambda + i\lambda_3 T_3)\cdot\psi \tag{10}$$

It is also invariant under two transformations of the type (8), but these transformations are not canonical, i.e., they do not preserve the commutation relations given, for example, in the book by Akhiezer and Berestetskii.[4]

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MARSHAK INVARIANCE AND FOUR-FERMION INTERACTIONS

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A classification of baryons and leptons which is useful for describing weak interactions is proposed. Marshak invariance is generalized to all baryons and leptons.

IT has been shown[1] that every known baryon can be characterized by a triplet of numbers t_1, t_2, t_3, which are determined as follows. We imagine that the baryons are located at the vertices of a unit cube (see Fig. 1). Then the numbers t_1, t_2, t_3, are the coordinates of the corresponding vertices in a Cartesian coordinate system.

Marshak[2] pointed out the interesting fact that weak interaction processes are invariant to the interchanges $p \longleftrightarrow \nu$, $n \longleftrightarrow e^-$, $\Lambda^0 \longleftrightarrow \mu^-$. One may conjecture that this invariance extends to all baryons.

We represent the set of leptons in the form of a unit cube like the baryon cube (Fig. 2). In order to keep the expression for electric charge, $q = e(t_2 + t_3)$, the same for baryons and leptons, one must assume that the cube is described in a left-handed coordinate system. In the case shown, we have two neutrinos, ν_1 and ν_2, which differ in the values of their leptonic charges $l = 2t_1$ (see also reference 3).

Marshak invariance can now be formulated in the statement that the set of weak interaction processes is invariant to interchange of the particles which are at the same cube vertex, if only those interchanges which conserve the number of baryons are allowed. Comparison of the lepton cube with the baryon cube gives the transformation

$$\begin{array}{llll} p \leftrightarrow \nu_1, & \Lambda^0 \leftrightarrow \mu^-, & \Sigma^0 \leftrightarrow \mu^+, & \Xi^- \leftrightarrow \bar{\nu}_1, \\ n \leftrightarrow e^-, & \Sigma^+ \leftrightarrow \nu_2, & \Sigma^- \leftrightarrow \bar{\nu}_2, & \Xi^0 \leftrightarrow e^+. \end{array} \qquad (1)$$

The requirement of baryon number conservation arises from $[\mathscr{H}, N] = 0$, where N is the baryon number operator and $\mathscr{H}$ is the Hamiltonian describing the weak interactions of baryons with leptons and baryons. In the case of baryon-lepton interactions, this requirement is automatically taken into account when the transformation (1) is applied to the equations written in the form $B + L \rightarrow B' + L'$, where B, B′ are baryons and L, L′ are leptons. For example, according to the rule, the equation $n \rightarrow p + e^- + \bar{\nu}_1$ should be written in the form $n + e^+ \rightarrow p + \bar{\nu}_1$, or $n + \nu_1 \rightarrow p + e^-$. The transformation (1) transforms the latter two equations into $e^- + \Xi^0 \rightarrow \nu_1 + \Xi^-$ and $e^- + p \rightarrow \nu_1 + n$ which can be rewritten as $n \rightarrow p + e^- + \bar{\nu}_1$ and $\Xi^- \rightarrow \Xi^0 + e^- + \bar{\nu}_1$.

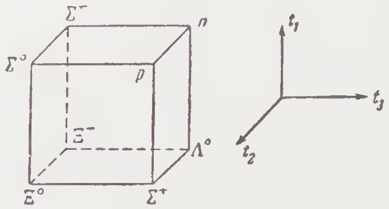

FIG. 1

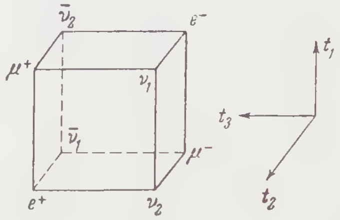

FIG. 2

For weak interactions involving antibaryons, it is necessary to apply the transformation (1) in its charge-conjugate form, i.e., $\bar{p} \longleftrightarrow \bar{\nu}_1$, etc.

The selection rules for weak interactions are now expressed as conservation of the three quantum numbers t_1, t_2, t_3 in weak interactions. With this rule, the process $\mu^+ \rightarrow e^+ + \bar{\nu}_1 + \nu_2$, for example, is allowed; the process $\mu \rightarrow 3e$ is forbidden.

The four-Fermion interaction Hamiltonian then has the isospin invariant form

$$H = \sum_k g_k \,|\, (\bar{\psi}_B T^{(k)} O \psi_B) + (\bar{\psi}_L T^{(k)} O \psi_L) \,|^2. \qquad (2)$$

Here ψ_B is the potential of the baryon field. (32-component spinor) and ψ_L is the potential of the lepton field. The matrices $T^{(k)}$ are the isotopic spin vectors[1], the numbers t_k are eigenvalues of the third components of these vectors. The 32-rowed matrix O determines the type of interaction. We note that the additional requirement that the Hamiltonian, H, be invariant to Fierz transformations gives $O = \Gamma^{\nu}(1 - \Gamma^{5})$, where $\Gamma^{\nu} = 1 \times \gamma^{\nu}$, $\Gamma^{5} = 1 \times \gamma^{5}$.

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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

ON THE E2 TRANSITION PROBABILITY FROM THE FIRST 2^+ LEVEL IN SPHERICAL NUCLEI

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The probability for a E2 transition from the first excited 2^+ state to the ground state as a function of the number of particles is derived on the basis of a microscopic description of the first excited 2^+ state in spherical nuclei.

It is well known that in spherical nuclei the first excited state with spin 2 has collective properties. In particular, the radiative transition probability to the ground state is on the average an order of magnitude larger than a single particle transition probability.

Recently there have appeared several papers[1-3] in which the first 2^+ level was treated assuming pairing and quadrupole forces. Then, for example, for the case of N particles in a level with angular momentum j the Hamiltonian has the form[3]

$$H = \frac{f_0}{2}\sum_{\mu} \overset{+}{A}_{2\mu} A_{2\mu} - \frac{F_2}{2}(uv)^2 \sum_{\mu} [\overset{+}{A}_{2\mu} - (-1)^{\mu} A_{2-\mu}][A_{2\mu} - (-1)^{\mu} \overset{+}{A}_{2-\mu}],$$

$$\overset{+}{A}_{2\mu} = \sum_{m,m'} (jjm - m'|2\mu)\, \overset{+}{\alpha}_m \overset{+}{\alpha}_{-m'}, \qquad (1)$$

where f_0 and F_2 are constants characterizing the strength of the pairing and the quadrupole force, $A^{+}_{2\mu}$ is the creation operator of a "phonon" of angular momentum 2 and projection of the angular momentum μ, and $(jjm - m'|2\mu)$ are Clebsch-Gordan coefficients.

The particle operators a_m are connected with the quasi-particle operators α_μ by the relation

$$a_m = u\alpha_m + v(-1)^{j+m} \overset{+}{\alpha}_{-m}, \qquad (2)$$

where

$$v = (N/2\Omega)^{1/2}, \qquad u = (1 - N/2\Omega)^{1/2}, \qquad \Omega = j + 1/2.$$

The Hamiltonian (1) can be diagonalized by introducing new operators

$$\overset{+}{B}_{2\mu} = \chi \overset{+}{A}_{2\mu} - (-1)^{\mu} \varphi A_{2-\mu}, \qquad (3)$$

$$\chi = (f_0 + 2E_2)/2\sqrt{2f_0E_2}, \qquad \varphi = -(f_0 - 2E_2)/2\sqrt{2f_0E_2}, \qquad (4)$$

$$E_2 = \frac{f_0}{2}\sqrt{1 - \frac{2N}{\Omega}\left(1 - \frac{N}{2\Omega}\right)\frac{E_2}{f_0}}. \qquad (5)$$

E_2 is the energy of the 2^+ level; its wave function can be written as

$$\Phi_{2^+} = \overset{+}{B}_{2\mu} |\Phi_0\rangle, \qquad (6)$$

where $|\Phi_0\rangle$ – ground state wave function.

Let us investigate the probability for the ground state transition for the simple case where N particles are in one level with an angular momentum $j \gg 1$. Then the operator for the quadrupole moment has the form

$$\hat{Q}_{2\mu} = -\langle j|r^2|j\rangle \sum_{m,m'} (-1)^{j-m} (jjm - m'|2 - \mu)\sqrt{\frac{2j+1}{5}}\, \overset{+}{a}_{m'} a_m, \qquad (7)$$

where $\langle j|r^2|j\rangle$ – radial matrix element. The reduced E2 transition probability has the form

$$B_{2\to 0}(E2) = \sum_{u} |\langle \Phi_0 | \sqrt{\frac{5}{16\pi}}\, \hat{Q}_{2\mu} | \Phi_{2+}\rangle|^2. \qquad (8)$$

Using (2), (3), (6) and (7) one finds easily

$$B_{2\to 0}(E2) = (2j/16\pi)\langle j|r^2|j\rangle^2 (uv)^2 (\chi - \varphi)^2. \qquad (9)$$

Inserting in (9) the expressions for u, v, χ and φ we obtain

$$B_{2\to 0}(E2) = \frac{\langle j|r^2|j\rangle^2}{16\pi} N\left(1 - \frac{N}{2\Omega}\right)\left[1 - \frac{2N}{\Omega}\left(1 - \frac{N}{2\Omega}\right)\frac{F_2}{f_0}\right]^{-1/2} \qquad (10)$$

or

$$B_{2\to 0}(E2) = \frac{\langle j|r^2|j\rangle^2}{16\pi} N\left(1 - \frac{N}{2\Omega}\right)\frac{f_0}{2E_2}. \qquad (11)$$

Equation (10) gives the dependence of $B_{2\to 0}$ on the number of particles in the shell.

Equation (11) shows more clearly the origin of the collective nature of the transition. The transition probability is proportional, first, to the number N of the particles in the shell and, second, to the ratio $f_0/2E_2$ which increases monotonically as the shell is being filled. Indeed, f/2 corresponds to the energy required to create two quasi-particles. It has a constant magnitude. On the other hand one sees from (5) that the energy E_2 decreases as the shell is being filled, up to the point where the shell is half full. It can be several times smaller than $f_0/2$. The last factor is associated with the quadrupole force. For $F_2 = 0$ the quantity $f_0/2E_2$ takes on its smallest value which equals unity.

The author expresses his gratefulness to S T. Belyaev for the opportunity to see his work prior to publication.

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GIANT QUANTUM OSCILLATIONS IN THE ACOUSTICAL ABSORPTION BY A METAL IN A MAGNETIC FIELD

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An expression is obtained for the acoustical absorption coefficient in metals in a magnetic field which satisfies the condition $\hbar\Omega \gg kT$ (Ω is the electron Larmor frequency) for the case when the acoustical wavelength is greater than the Larmor radius of the conduction electrons, but significantly less than their mean free path. It is found that under such conditions the absorption coefficient oscillates as a function of 1/H. The nature of these oscillations is the same as in the de Haas—van Alphen or Shubnikov—de Haas effects. However, the oscillations can under certain conditions be gigantic, i.e., they are not a small correction to a non-oscillatory part of the acoustical absorption coefficient, but are such that the maximum value of the absorption coefficient far exceeds the minimum.

AT low temperatures, when the mean free path l of conduction electrons in a metal is sufficiently large, the absorption of sound can be considered as the direct absorption of phonons by the electrons of the metal. In the absence of a magnetic field the major role in the absorption is taken by electrons whose velocity v_κ in the direction of the acoustical wave vector κ is equal to the phase velocity of sound w_p.

This fact was pointed out by Akhiezer, Kaganov and Lyubarskii.[1] It is easy to satisfy oneself that, by considering the conservation laws during phonon absorption by the electron

$$\varepsilon(\mathbf{p}) + \hbar\omega_\varkappa = \varepsilon(\mathbf{p} + \hbar\boldsymbol{\varkappa}), \qquad (1)$$

where $\mathbf{p}$ is the quasi-momentum of the electron, ϵ is its energy, $\omega = w_p\kappa$ is the frequency of sound, and by expanding $\epsilon(\mathbf{p} + \hbar\kappa)$ to terms of first order in the small quantity κ

$$\varepsilon(\mathbf{p} + \hbar\boldsymbol{\varkappa}) = \varepsilon(\mathbf{p}) + \hbar v_\varkappa \varkappa, \qquad (2)$$

we obtain the condition

$$v_\varkappa = w_p, \qquad (3)$$

which has been mentioned above. Electrons with a different velocity component in the direction of the vector κ can participate in the absorption only in so far as they interact with scatterers. As was shown in reference 1, if $\kappa l \gg 1$ their contribution to the absorption is of order $1/\kappa l$ compared with the contribution of electrons whose velocity satisfies condition (3).

An analogous situation also occurs in the presence of a magnetic field. We shall consider for simplicity the case of a quadratic isotropic dispersion law for electrons; the energy of an electron in the magnetic field then takes the form

$$\varepsilon_n = \hbar\Omega\,(n + 1/2) + p_z^2/2m, \qquad (4)$$

where n is the Landau oscillator quantum number, $\Omega = eH/mc$ is the Larmor frequency, m is the effective mass, and p_z is the component of momentum in the direction of the magnetic field **H** which is along the z axis. Taking into account the conservation of energy and of the z component of the quasi-momentum, we have

$$\hbar\Omega\,(n + 1/2) + p_z^2/2m + \hbar\omega_\varkappa = \hbar\Omega\,(n' + 1/2) + (p_z + \hbar\varkappa_z)^2/2m. \qquad (1a)$$

Putting

$$\varepsilon_{n'}(p_z + \hbar\varkappa_z) = \varepsilon_{n'}(p_z) + \hbar v_z \varkappa_z, \qquad (2a)$$

we obtain the condition

$$\Omega\,(n' - n) + w_p\varkappa = v_z\varkappa_z. \qquad (5)$$

If the magnetic field is so large that

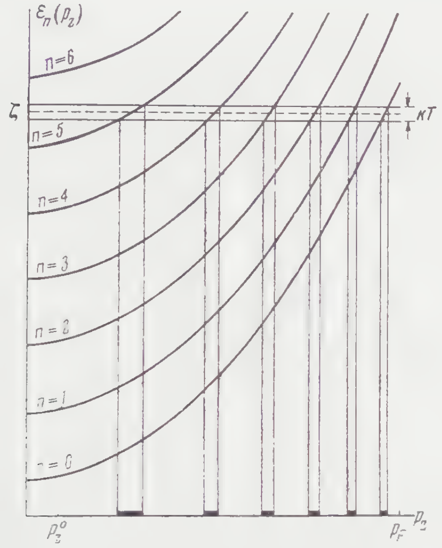
$$\Omega > \varkappa_z v_F \qquad (6)$$

(v_F is the Fermi velocity), then condition (5) can only be satisfied for n = n′. Consequently, the quasi-momentum of the electrons which participate in the acoustical absorption in a magnetic field is equal to

$$p_z^0 = mw_p / \cos\vartheta, \qquad (7)$$

where θ is the angle between the vectors κ and H. But $w_p \ll v_F$, therefore, if this angle differs sufficiently from a right angle, most of the absorption is due to electrons with small values of p_z, far smaller than the limiting Fermi momentum p_F.

On the other hand, it is obvious that only electrons with energies lying in the interval where the Fermi distribution is changing can participate in absorption. In order to see more clearly the values of p_z which correspond to these energy values, we turn to the figure. In it are shown the parabolae which give the variation of the electron energy with p_z for the first few values of n. The parabolae intersect the strip of width kT, the center of which is at the chemical potential ζ. The width of the strip is smaller than the distance between the parabolae, which corresponds to the condition



$$\hbar\Omega \gg kT. \qquad (8)$$

By projecting onto the abscissa axis the portions of the parabolae cut off by the strip, we find that, when condition (8) is satisfied where the Fermi distribution is changing, intervals of allowed and forbidden values of p_z exist. The former intervals are indicated in the Figure by thick lines; for the corresponding values of p_z the derivative with respect to ζ/kT of the Fermi function is of the order of unity. The latter intervals are indicated by thin lines on the abscissa axis, and for the corresponding values of p_z the derivative with respect to ζ/kT of the Fermi function is exponentially small. The position of these intervals depends on the magnetic field H, because the distance between the parabolae in the figure changes with changing H. If the value of H is such that the momentum p_z^0 is in the range of allowed values of p_z, strong acoustical absorption by free electrons occurs. For a different value of H the momentum p_z^0 can lie in a range of forbidden values of p_z, and then acoustical absorption occurs only because of the scattering of conduction electrons, i.e., it is much smaller than in the first case. The general situation is thus one in which giant oscillations in the acoustical absorption coefficient occur provided that

$$\zeta \gg \hbar\Omega \gg kT. \qquad (9)$$

We note that the other kinetic coefficients, such as the conductivity, also oscillate when (9) is satisfied. The occurrence of these quantum oscillations is also connected with the presence of allowed and forbidden intervals of p_z on the Fermi surface. However, these oscillations are a small correction — usually of order $(\hbar\Omega/\zeta)^{1/2}$ — to a major effect, and can be calculated using classical theory.

In our case giant quantum oscillations occur. Therefore, when calculating the energy absorbed, which we now proceed to do, it is necessary to use quantum theory. We neglect the contribution to the absorption from transverse electric fields arising during deformation of the conductor by the sound. (The question of whether it is permissible to neglect this has been considered previously.[2,3]) Then the energy operator for the interaction of the electron with the sound wave has the form

$$V = \tfrac{1}{2}(Ue^{-i\omega t} + U^+ e^{i\omega t}), \qquad (10)$$

where $U = \Lambda_{ik} u_{ik} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}$. Here u_{ik} is the tensor amplitude of the deformation in the sound wave, Λ_{ik} is the tensor calculated in reference 1, the components of which in general can depend on the components of the quasi-momentum operator. Strictly, the quantity $\Lambda_{ik}u_{ik}$, being the difference between the deformation and electrostatic potentials, should itself oscillate in the magnetic field. However, it is not difficult to see that the relative size of these oscillations is small; therefore they will be ignored when considering the giant oscillations in the absorption.

We consider the scattering of electrons to be elastic, which is permissible in most cases. Then the energy density absorbed by the electrons in unit time is

$$q = -\frac{1}{V_0}\,\frac{\pi}{2\hbar} \times \sum_{aa'} (F_a - F_{a'})\,\hbar\omega_{aa'}\,|\langle a|U|a'\rangle|^2\,\delta(\hbar\omega_{aa'} + \hbar\omega). \qquad (11)$$

Here a and a′ denote the states of the electron in the field of the scatterers in the presence of the external magnetic field, $\hbar\omega_{aa'} = \epsilon_a - \epsilon_{a'}$, F_a is the Fermi function, $<a|v|a'>$ is the matrix element of the operator U, V_0 is the volume of the crystal.

If $\hbar\omega \ll kT$, we can put

$$F_a - F_{a'} = (\partial F_a / \partial\zeta)\, \hbar\omega_{a'a}| = (\partial F_a / \partial \zeta)\, \hbar\omega. \tag{12}$$

The absorption coefficient Γ is obtained by dividing equation (11) by the flux density of acoustical energy equal to

$$\rho\omega^2 u_0^2 w / 2, \tag{13}$$

where ρ is the density of the crystal, u_0 is the amplitude of the oscillations in the sound wave, w is the group velocity of sound. In order to obtain the zero-order approximation we will neglect the general effect of the scatterers on the absorption of sound. To do this it suffices to replace the indices a and a′ in (10) by α and α', respectively, which characterize the free electron states in a magnetic field. The quantity α is the totality of the quantum numbers n, p_z, X, and s_z, where $\hbar s_z/2$ is the spin component in the direction of the magnetic field, X is the coordinate of the center of the Landau oscillator. After this the following expression is obtained for the acoustical absorption coefficient:

$$\Gamma = \frac{\pi}{V_0 \rho u_0^2 w} \sum_{\alpha\alpha'} \frac{\partial F_\alpha}{\partial \zeta} \langle \alpha | U | \alpha' \rangle |^2 \delta(\omega_{\alpha\alpha'} + \omega). \tag{14}$$

For simplicity we now consider the idealized case where the components of the tensor Λ_{ik} are essentially constant, the electron spectrum is quadratic and isotropic, and the vectors $\boldsymbol{\kappa}$ and **H** are parallel to one another. Then

$$\langle \alpha' | U | \alpha \rangle = \Lambda_{ik} u_{ik} \delta_{s_z s_z'} \delta_{nn'} \delta_{XX'} \delta_{p_z + \hbar\varkappa_z, p_{z'}} \tag{15}$$

and we have for the coefficient Γ

$$\Gamma = \Gamma_0 \frac{\hbar\Omega}{8kT} \frac{\varkappa}{m} \sum_{n, s_z} \int dp_z\, \delta\left(\frac{\varkappa p_z}{m} + \frac{\hbar\varkappa^2}{2m} - \omega\right)$$

$$\times \mathrm{ch}^{-2}\left[\frac{\zeta - \hbar\Omega(n + {}^1\!/_2) - \mu_0 s_z H - p_z^2/2m}{2kT}\right]. \tag{16}^*$$

Here μ_0 is the Bohr magneton,

$$\Gamma_0 = m^2 |\Lambda_{ik} u_{ik}|^2 / 2\pi\hbar^2 \rho u_0^2 \varkappa w \tag{17}$$

is the acoustical absorption coefficient when H = 0, obtained by Akhiezer et al.[1]

After integrating over p_z, we obtain

$$\Gamma = \Gamma_0 \frac{\hbar\Omega}{8kT} \sum_{n, s_z} \mathrm{ch}^{-2}\left[\frac{\hbar\Omega(n + {}^1\!/_2) + s_z\mu_0 H - \zeta}{2kT}\right]. \tag{18}$$

In deriving (18) we have neglected in the argument of the hyperbolic cosine the small quantity $(p_z^0)^2/2m$, which can only cause an insignificant overall shift of the oscillatory picture.

For $\hbar\Omega \ll kT$ the sum over n in (18) can be replaced by an integral which is easily evaluated and gives

$$\Gamma = (\Gamma_0/2) \int_0^\infty \frac{dy}{\mathrm{ch}^2(y - \zeta/2kT)} \approx \Gamma_0.$$

This expression for Γ agrees with that obtained previously.[3]

When $\hbar\Omega \gg kT$ the graph of the function $\Gamma(1/H)$ shows a series of sharp oscillatory maxima, the distance between which is

$$\Delta H^{-1} = e\hbar/mc\zeta, \tag{19}$$

and the height of which is proportional to H, separated by wide, gently sloping minima, the heights of which are exponentially small.

Equation (19) can only describe the behavior of the absorption coefficient close to the maxima. In order to obtain the value of the absorption coefficient for the minima, and at the same time to estimate the limits of applicability of (19), it is necessary to obtain the correction to it due to scattering. To do this we replace the δ function in (16) by the expression

$$\frac{1}{\pi} \frac{\nu}{\nu^2 + (\hbar\varkappa p_z/m + \hbar\varkappa^2/2m - \omega)^2}. \tag{20}$$

Such a replacement corresponds to the assumption that a relaxation time $t_0 = 1/\nu$ exists, and yields the order of magnitude of the necessary correction. In the classical theory (see reference 2) when $\kappa l \gg 1$ a relaxation time exists for most scattering mechanisms. The proof that a relaxation time exists when $\kappa l \gg 1$ in the quantum region presents well-known difficulties, and will be given for some cases in a separate paper.

We consider the case of greatest interest as regards possible experiments: $\hbar\kappa^2/2m \ll \nu$ and $\omega \ll \nu$. The expression (20) can then be taken as equal to

$$\nu/\pi\,[\nu^2 + (\hbar\varkappa p_z/m)^2]. \tag{21}$$

We replace the δ function in (16) by (21) and transform to the dimensionless integration variable $y = p_z(2mkT)^{-1/2}$. Then (16) becomes

$$\Gamma = \Gamma_0 \frac{\hbar\Omega}{2kT} \int dy \frac{1}{\pi} \frac{B}{1 + B^2 y^2} \sum_{n, s_z} \frac{1}{4} \mathrm{ch}^{-2}\left(\frac{y - A_n}{2}\right). \tag{22}$$

Here $B = (2kT/m)^{1/2}\varkappa/\nu$; $A_n = [\zeta - \hbar\Omega(n + {}^1\!/_2) - s_z\mu_0 H]/kT$.

The expression under the integral in (22) contains the product of two rapidly-changing functions. The maximum of the first is at y = 0 and has a width $1/B$; far from the maximum it falls off as $1/\pi By^2$. The second function consists of a system of peaks, the width of each one being of order unity. The position of the peaks changes as the magnetic field changes, and the distance between them far exceeds their width when $\hbar\Omega/kT \gg 1$. Far from the peaks the second function decreases exponentially: $\exp(-|A_n - y^2|/2)$.

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*ch = cosh.

We shall treat the case when the distance between the two peaks of the second function closest to the maximum of the first function greatly exceeds the width of this maximum. This condition is equivalent to the inequality

$$B(\hbar\Omega/kT)^{1/2} \gg 1, \text{ or } \varkappa l(\hbar\Omega/\zeta)^{1/2} \gg 1. \tag{23}$$

Now, if the value of the magnetic field is such that the position of one of the peaks of the second function coincides with the maximum of the first, then a giant oscillation occurs. If the width of the maximum of the first function is much smaller than the width of the peak of the second, i.e.,

$$B \sim \varkappa l(kT/\zeta)^{1/2} \gg 1, \tag{24}$$

the first function can be replaced by a δ function. Thus, under condition (24), formula (18) accurately describes the shape of the absorption line close to a maximum. In the case $B \lesssim 1$ the shape of the absorption line is different.

The absorption coefficient has a minimum when two closest peaks of the second function are symmetrically situated with respect to the maximum of the first. In this case the first function changes slowly in the region of both peaks; therefore the peaks themselves can be approximated by δ functions. The evaluation of the coefficient Γ at a minimum then presents no difficulty, and gives the following ratio of the maximum value of the absorption coefficient to the minimum:

$$\Gamma_{max}/\Gamma_{min} \sim \varkappa l\,(\hbar\Omega/\zeta)^{1/2}\hbar\Omega/kT \gg 1, \tag{25}$$

i.e., the oscillations are, in fact, gigantic. The estimate (25) retains its significance also when $B \sim 1$, and gives

$$\Gamma_{max}/\Gamma_{min} \sim (\hbar\Omega/kT)^{3/2}. \tag{26}$$

The case $B \ll 1$ is not of great interest, because it is difficult to satisfy simultaneously in an experiment this inequality and (23).

Giant oscillations of the acoustical absorption coefficient can also occur for an arbitrary shape of the Fermi surface and for arbitrary form of the deformation potential, provided that the motion of electrons with those values of p_z which make $\langle\alpha|v_z|\alpha\rangle$ tend to zero is finite. In fact, in this case the matrix elements of the deformation potential (neglecting spin-orbit interaction) are

$$\langle\alpha|U|\alpha\rangle = \langle n_\alpha|\Lambda_{ik}u_{ik}|n_\alpha\rangle\,\delta_{s_z s_z'}\,\delta_{p_z+\hbar\varkappa_z, p_z'}\,\delta_{XX'}. \tag{27}$$

But if $\Omega > \kappa_z v_F$, then $n = n'$, and we conclude that all the previous estimates of the coefficient Γ at a maximum and a minimum remain valid. However, the whole oscillatory picture can be very complicated, because there can exist several values of p_{zm} for which $\langle\alpha|v_z|\alpha\rangle$ is close to zero. The number of these values can be determined from the number of different periods on the curve of $\Gamma(1/H)$.

As in other well-known cases (see, for example, the work of Lifshitz and Kosevich[4]), the period of the quantum oscillations in our case is related to the area $S(\epsilon, p_z)$ cut from the constant energy surface by the plane $p_z = p_{zm}$ by the formula

$$\Delta H_m^{-1} = 2\pi e\hbar/cS(\varepsilon, p_{zm}). \tag{28}$$

However, the essential difference is that the periods of the oscillations in our case are determined not by the extremal sections, but by the sections involving those $p_z = p_{zm}$ which make $\langle\alpha|v_z|\alpha\rangle$ tend to zero.

Clearly the giant oscillations can be smeared out if the specimen has an even relatively weak polycrystallinity. If this is to be avoided, the angle φ describing the spread in orientation of the crystallites which constitute the specimen must satisfy the condition

$$\varphi \ll \hbar\Omega/\zeta. \tag{29}$$

This inequality is satisfied more easily, the greater the magnetic field and the smaller the value of ζ. The giant oscillation effect is, therefore, most easily observed in semimetals and semiconductors with Fermi degeneracy.

Thus, the classical theory of acoustical absorption in a magnetic field previously developed[2,3] is, generally speaking, only applicable when $\hbar\Omega \ll kT$. If the reverse inequality applies, quantum theory must be used.

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ELECTRODYNAMICS OF A ZERO MASS SPINOR PARTICLE

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The electrodynamics of a massless two-component particle with charge e_1 is considered. Despite a number of peculiarities the theory turns out to be not internally inconsistent and is as complete as ordinary electrodynamics. A rough experimental estimate gives $e_1 \lesssim 10^{-8}$ e. The possibility that the known neutrinos carry such a charge e_1 is discussed.

1. INTRODUCTION

SIMPLICITY of the basic equations is an important consideration in attempts to describe the properties of elementary particles. As examples we might cite Dirac's equation, which is the simplest of all equations describing particles with half-integral spin, the two-component neutrino theories, or the universal "two-component" weak interaction.[1] It is likely that considerations of simplicity and symmetry will continue to be taken into account in the future development of the theory. In such a framework it becomes quite attractive to consider the possibility of existence of a charged, massless, two-component particle, described by the Lagrangian

$$L(x) = -u^{+}\sigma_r(p_r - e_1 A_r)u. \qquad (1)$$

Here $\sigma_r = (\sigma, 1)$, and $u(x)$ is a two-component spinor. Such a possibility is not quite obvious, however, because many of the formulas of electrodynamics contain the mass of the particle in the denominator or as the argument of a logarithm. This then raises the question of consistency of electrodynamics for a massless particle, which might be of interest for various reasons.

1. It might be useful to trace the role played by the mass in electrodynamics: are the mass and charge related in the existing theory, is there an "electromagnetic" mass?

2. For the development of the theory of particles and interactions it will, no doubt, be useful to know the properties of permissible "solutions," particularly the simplest solutions.

3. Finally, if such a theory is internally consistent one might raise the question about the existence of such particles. It is obvious that the interaction (1) is impossible with the constant $e_1^2 = e^2 = 1/137$. However it is not out of the question that a two-component massless particle (neutrino), characterized by the weakness of its interaction, has a weak interaction with photons.

2. CLASSICAL ELECTRODYNAMICS*

The Lorentz equation contains no mass; the total energy provides a "measure of inertia" and consequently the equations of motion in an external field are valid for $m = 0$ too.

The peculiarities in the interaction with radiation have to do with the equality of the speed of the particle and the speed of the radiation field. A massless particle does not fall behind the electromagnetic wave, produced by a change in the particle velocity, so that it is not possible to make a strict division of the field into the particle field and radiation field. A single spherical wave is produced, with a singularity at the point occupied by the charge. The proper field may be determined only in the vicinity of the particle, where it is legitimate to neglect the curvature of the front of the spherical wave. The electric and magnetic fields are here given by the formulas

$$\mathbf{E} = e_1\delta(z - t)\,\boldsymbol{\rho}/\rho^2, \qquad \mathbf{H} = [\mathbf{v}\mathbf{E}], \qquad (2)\dagger$$

where the z axis of the cylindrical coordinate system (z, ρ) has been chosen along the direction of the particle velocity $\mathbf{v}$. The expression (2) represents the limiting case of the well known relativistic contraction of the proper fields as a consequence of motion.

The formulas for radiation become, for $m = 0$, logarithmically divergent for small angles θ between the directions of radiation and the particle velocity. For example in the radiation problem in

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*By classical electrodynamics we mean, as is usual, the interaction with radiation of low frequency $\omega \ll \varepsilon$, where ε is the particle energy. We use units such that $h = c = 1$, and a metric such that $ab = a_r b_r = \mathbf{a}\cdot\mathbf{b} - a_0 b_0$.

†$[\mathbf{v}\mathbf{E}] = \mathbf{v} \times \mathbf{E}$.

β decay (up to $t = 0$ the charge is at rest and thereafter moves with velocity $\mathbf{v}$) the spectral energy density of radiation $I(\omega, \theta)$ is given by

$$I(\omega, \theta)\, d\omega d\theta = \frac{e_1^2}{2\pi} v^2 \frac{\sin^3\theta\, d\omega d\theta}{(1 - v\cos\theta)^2}, \tag{3}$$

so that for $v = 1$ the expression $I(\omega) = \int I(\omega, \theta)\, d\theta$ is logarithmically divergent. The divergence arises from an incorrect application of the formula, correct for the fields at a distance from the charge, at the position of the charge $\theta = 0$. If the radiation for small θ is determined as the energy emitted in that direction after the particle was deflected, then the quantity $I(\omega)$ will depend logarithmically on the time T spend by the particle in moving in the given direction:

$$I(\omega) = \frac{e_1^2}{4\pi^2} \int \frac{[\mathbf{nv}]^2}{(1 - \mathbf{nv})^2} \left| \exp\{i(\omega - \mathbf{kv})T\} - 1 \right|^2 \approx \frac{2e_1^2}{\pi} \ln \omega T. \tag{4}$$

Consequently the particle is not absolutely stable, its energy and momentum going eventually over into radiation. A semi-classical estimate for the dependence of the particle energy on time, due to the losses in such a "forward radiation," results in the formula

$$\varepsilon(t) = \varepsilon(0) \exp\{-(e_1^2/2\pi) \ln \varepsilon(0)t\}, \qquad \varepsilon(0)t \gg 1. \tag{5}$$

This problem is considered in detail in Sec. 4. It is clear that for $e_1^2 \ll 1$ the transformation of the energy into radiation is extremely slow and the particle is metastable, immeasurably more stable than, say, the pion or neutron. It is also clear from physical considerations that any deviation of the trajectory from a straight line violates the conditions for coherent interference and the difference between the radiations of massive and massless particles disappears. Thus, the classical expressions for the radiation of an ultrarelativistic electron moving in a magnetic field[2]* or in a material medium,[3] do not contain the mass and are therefore directly applicable to our case.

3. THE DIRAC EQUATION. PRESERVATION OF MASSLESSNESS. FORBIDENNESS OF DIRECT DECAYS

The Dirac equation becomes in our case

$$\sigma(p - e_1 A)\, u(x) = 0. \tag{6}$$

The positive and negative frequency solutions of Eq. (6) go over into each other continuously. Bound states in an external field, vanishing at infinity, do not exist; quasistationary states are, of course, possible. The energy levels in a constant magnetic field $H = H_Z$ are given by $\epsilon_n = [p_Z^2 + 2e_1 H(n + 1)]^{1/2}$, where p_Z is the z component of the momentum and n is the number of the state. In this way the field H leads to a gap between states with opposite signs of the energy, with the quantity $(2e_1 H)^{1/2}$ acting as a mass.

The quantization is carried out in the usual way. The interaction Lagrangian becomes then

$$L(x) = e_1 j_r(x) A_r(x) = e_1 N(u^+(x)\sigma_r u(x)) A_r(x). \tag{7}$$

In order to derive our case from the formulas of conventional electrodynamics in the limit $m = 0$, we observe that the operator $u(x)$ can be obtained from the full Dirac field $\psi(x)$ by simple projection: $u(x) = \frac{1}{2}(1 - \gamma_5)\psi(x)$. Since γ_5 commutes with any pair of matrices γ_r it is sufficient to insert $\frac{1}{2}(1 - \gamma_5)$ once in each electron line of a Feynman diagram.

It is clear from the formulas of electrodynamics that the electromagnetic mass δm is proportional to the bare mass m_0,[4] so that if $m_0 = 0$ then also $\delta m = 0$. Thus the masslessness is preserved and the singularity of the Green's function $G(p)$, as in the free field case, lies at the point $p^2 = 0$ [see Eq. (15)].

Unlike the electron case, the decay of the massless particle into a photon and a particle moving in the same direction is allowed by the conservation laws. Furthermore the statistical weight is not equal to zero since any division of the energy between the decay products is possible. However, as is easy to see from Eq. (7), the matrix element for the transition vanishes. This has to do with the transverse nature of the photon and conservation of the projection of the angular momentum along the direction of motion: for the photon this quantity is always +1 or −1, for the particle it is always ½ and therefore the decay is forbidden. For the same reason a photon cannot produce a pair of these particles since in the final state the spin projection on the direction of motion is zero. And since the decay of one particle into three is already forbidden by statistical weight factor it follows that no direct decays occur.

Nevertheless we do not have complete stability, as is already known from classical theory. Higher order perturbation theory approximations contain logarithmic divergences at small θ, analogous to the classical Eqs. (3) and (4). The situation is very much the same as the well known "infrared catastrophe." The divergence indicates the inapplicability of perturbation theory to ultra small angles, as well as ultra small frequencies. For small e_1^2 this divergence, like the infrared divergence, is of no practical importance since all angular integrals must be cut off at $\theta_{\min} \sim \Delta\theta$ (just like integrals over frequency are cut off at $\omega_{\min} \sim \Delta\omega$), where $\Delta\theta$ is the experimental error in the selection of photons emitted at small angles to the beam direction. However, just as in the case of the infrared catastrophe, the problem can be solved without making use of perturbation theory, and the divergence indicates in both cases the basic instability in the problem.

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*The classical formulas for the spectrum of radiation density are valid only for frequencies $\omega \lesssim \varepsilon$. Consequently the expressions given by Landau and Lifshitz[2] for the total radiation intensity in a magnetic field are not applicable here.

4. METASTABILITY OF MASSLESS PARTICLE AND PHOTON

As an example we discuss the long wavelength radiation in β decay. Our special case $v = 1$ can be obtained from known solutions of the infrared problem. A simultaneous discussion of the singularities encountered in the emission of "soft" photons and in the "forward emission" will serve to emphasize the similarity of these phenomena.

Suppose that at $t = 0$ a particle with momentum $\mathbf{p}$ is produced. Then in perturbation theory we get for the amplitude of the electron + photon state

$$c_{\mathbf{p}'\mathbf{k}\lambda}(t) = V_{\mathbf{p}'\mathbf{k}\lambda,\,\mathbf{p}}(\varepsilon_p - \varepsilon_{p'} - k)^{-1}$$
$$\times [\exp\{i(\varepsilon_{p'} + k - \varepsilon_p)t\} - 1]. \tag{8}$$

Here $\mathbf{p}' = \mathbf{p} - \mathbf{k}$ and $V_{\mathbf{p}'\mathbf{k}\lambda,\,\mathbf{p}}$ is the matrix element for the transition. The exponential term in Eq. (8) describes the proper electromagnetic field of the particle,[5] whereas the other term describes the radiation. For large t, when $(\epsilon_{p'} + k - \epsilon_p)t \approx (k - \mathbf{k}\cdot\mathbf{v})t \gg 1$, these two parts of the field separate since in physical expressions, bilinear in $c_{p'k\lambda}$, the oscillating interference term drops out. Consequently the probability of radiation is given by the expression (for simplicity we restrict ourselves to the interaction with transverse photons)

$$W = \sum_{\mathbf{k}\lambda} \left| \frac{V_{\mathbf{p}'\mathbf{k}\lambda,\,\mathbf{p}}}{\varepsilon_{p'} + k - \varepsilon_p} \right|^2 = \frac{e_1^2}{4\pi^2} \int \frac{d\mathbf{k}}{k} \, \frac{v^2 - (\mathbf{nv})^2}{(k - \mathbf{kv})^2}. \tag{9}$$

For small k the quantity W diverges, and for $v = 1$ it also diverges for small angles between $\mathbf{k}$ and $\mathbf{v}$. The same divergence appears in the norm of the stationary state particle + photon, as calculated in perturbation theory. At the same time the quantity $\Sigma\,|c_{p'k\lambda}(t)|^2$, as calculated from Eq. (8), is finite but grows with time (for $v < 1$ as $\ln t$, for $v = 1$ as $\ln^2 t$). Consequently the calculation of the radiation probability from Eq. (9) is valid at the instant t only if $(k - \mathbf{k}\cdot\mathbf{v})t \gg 1$, since otherwise the appropriate photons have not yet separated into "real" and "virtual" ones. The total transition probability from the state with no photons does not approach a constant for large values of t, but increases as a consequence of the continuous creation of photons with $(k - \mathbf{k}\cdot\mathbf{v})t \gg 1$. Therefore the perturbation theory Eqs. (8) and (9) are valid only for small t or $(k - \mathbf{k}\cdot\mathbf{v})^{-1}$.

The complete solution of the problem may be taken from the work of Glauber.[6] If the state vector is written as $\Phi(t) = S(t, t_0)\Phi(t_0)$ then in the region of soft photons under consideration the matrix $S(t, t_0)$ is given, accurate up to a phase, by*

$$S(t, t_0) = \exp\left\{ i \int_{t_0}^{t} \mathbf{j}(\mathbf{x}, \tau)\mathbf{A}(\mathbf{x}, \tau)\,d\mathbf{x}\,d\tau \right\}, \tag{10}$$

where $\mathbf{j}(\mathbf{x}, \tau)$ is the particle current and is a prescribed function of $\mathbf{x}$ and τ, and $\mathbf{A}(\mathbf{x}, \tau)$ is the electromagnetic field operator in the interaction representation. Equation (10) corresponds to a Poisson photon distribution: the probability of emission up to the instant t of n photons in a given interval k is given by the formula

$$W_n(t) = \frac{[\bar{n}(t)]^n}{n!} \exp[-n(t)], \tag{11}$$

where, in our problem,

$$\bar{n}(t) = \frac{e_1^2}{4\pi} \int \frac{d\mathbf{k}}{k} \, \frac{v^2 - (\mathbf{nv})^2}{(k - kv)^2} \left| e^{i(k - \mathbf{kv})t} - 1 \right|^2. \tag{12}$$

We see that the expression $\Sigma\,|c_{p'k\lambda}(t)|^2$, with $c_{p'k\lambda}$ given by Eq. (8), indeed represents the average total number of emitted photons and may increase with t without bound. The amplitude of the photonless state $S_{00}(t) = \exp[-\bar{n}(t)]$ is damped out in time. The nonstationary formulation of the problem is here, of course, immaterial; in a stationary problem t would be replaced by R — the distance from the source or scatterer. Consequently, from the point of view of quantum mechanics, the massless particle is unstable to the same degree as the electron.

Equations (10) — (12) are valid for $k \ll \epsilon$. It is obvious that the physical picture of a logarithmic piling up of photons with time is also valid for $k \lesssim \epsilon$, although $W_n(t)$ does not for such frequencies have the Poisson form (11) since the considerable recoil causes the emissions to be correlated. For $k > \epsilon$ the singularities in the denominators in

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*It is easy to verify that Glauber's result, Eq. (10), is valid for arbitrary t_0 and t, and not just for the values $t_0 = -\infty$, $t = \infty$ considered by Glauber.[6]

Eqs. (8) and (9) disappear and perturbation theory is valid. It can be seen from Eqs. (10) and (11) that as the particle is deflected the "forward emission" photons are "shaken off" and the process starts all over again. It is also seen that both the infrared and the angular singularities disappear as a result of multiple scattering, magnetic fields, or any other effect leading to a nonrectilinear trajectory, since the coherent interference length $l(\mathbf{k}) = (k - \mathbf{k}\cdot\mathbf{v})^{-1}$ is now limited by the characteristic length l_{eff} of the process: $l(\mathbf{k}) \leq l_{\mathrm{eff}}$.

Let us indicate the manner in which the just described instability is reflected in the structure of the Green's function. The Fourier component

$$G(\mathbf{p}, \tau) = (2\pi)^{-1} \int_{-\infty}^{\infty} G(\mathbf{p}, p_0) \exp(-ip_0\tau)\, dp_0$$

is proportional to the amplitude for the probability of finding the initial state $\Phi_{0\mathrm{p}} = a_{\mathrm{p}}^{+}|\mathrm{vac}\rangle$ in the physical state $\Phi_{\mathrm{p}}(\tau) = \exp(-iH\tau)\Phi_{0\mathrm{p}}$, which evolves from $\Phi_{0\mathrm{p}}$ in the time τ.[7,8] A pole in $G(p)$ at the point $p_0 = \epsilon_{\mathrm{p}}$ corresponds to a stable particle, so that for large τ one has $G(\mathrm{p}, \tau) \sim \exp(-i\epsilon_{\mathrm{p}}\tau)$. In electrodynamics, however, owing to the fact that the continuous spectrum of the electron + photon states begins at the one-electron state, the pole of G becomes a branch point:

$$G(\mathbf{p}, p_0) \rightarrow \mathrm{const}\,(p_0 - \varepsilon_p)^{-1+\beta}, \qquad p_0 \rightarrow \varepsilon_p. \tag{13}$$

This indicates a damping of the initial state with time according to $G(\mathrm{p}, \tau) \sim \exp(-i\epsilon_{\mathrm{p}}\tau - \beta\, ln\, \epsilon_{\mathrm{p}}\tau)$. The quantity β can be found with the help of Eqs. (11) and (12), since $S_{00}(\tau)$ is also proportional to the amplitude of the photonless state. A covariant treatment with longitudinal and scalar photons taken into account[9,10] merely changes the quantity β in Eq. (13). For a massless particle the continuous spectrum again touches the pole so that the pole in G becomes a branch point. For purely transverse photons the singularity in $G(\mathrm{p}, p_0)$ is, according to Eq. (12), somewhat more complex than in the electron case. When however photons of all four polarizations are included even that difference disappears: it is clear from Eq. (15), Sec. 5, that the singularity in G has the form (13).

For purposes of exposition we have used above noncovariant expressions [for example Eq. (10)]. It is easy to pass to a covariant description. In the infrared catastrophe problem this is accomplished by introducing a small "photon mass" λ.[11] After addition of the probabilities of production of the particle with and without photons the quantity λ drops out of the answer, which is simply a result of unitarity — the total probability that the particle will be in one state or the other is conserved. These methods may be generalized so as to be applicable in our case and the quantity λ will disappear from the final answer after summing over the frequency interval $\omega < \omega_{\min}$ and the angular interval $\theta < \theta_{\min}$.

Up to now we were concerned with the change in time of the state of the particle. It follows from the considerations on the spectra and also simply from perturbation theory that a photon will also be unstable and will dissociate itself in time into a pair of massless particles moving in the same direction. In the photon Green's function too the pole changes into a branch point, again indicating the damping of the "bare" photon state and the appearance of the probability of pair formation. Under scattering by an external field one of the particles of the pair is deflected, and perturbation theory gives a probability for the emission of the other particle in the direction of motion of the photon that is logarithmically divergent with angle. In the absence of other physical reasons this divergence can be cut off by making use of the relations

$$(\varepsilon_p + \varepsilon_{\mathbf{k}-\mathbf{p}} - k)t \sim \frac{kp}{k-p}\frac{\theta^2}{2}t \gtrsim 1, \qquad \theta_{min} \gtrsim \left(\frac{\lambda\!\!\!^{-}}{R}\right)^{1/2}, \tag{14}$$

i.e., by making use of the finiteness of the time available for coherent interference in pair formation.

This then is the general picture of instability and mutual transformations if a massless particle exists. Let us emphasize once more that this lack of stationary states is only of importance in principle. For $e_1^2 < e^2$, and for reasonable values of $\lambda\!\!\!^{-}$, the distances R, at which the probability for "decay into emptiness", which is of order $e_1^2 \ln(R/\lambda\!\!\!^{-})$, becomes comparable with unity are many times in excess of any acceptable dimensions. At the same time the presence of any material medium will lead to multiple scattering and the divergence will disappear.

5. GREEN'S FUNCTION, VERTEX PART, RENORMALIZATIONS

The basic functions of electrodynamics may be obtained by taking the ultraviolet asymptote $p^2 \gg m^2$ of conventional electrodynamics.[12,13] Taking for simplicity the longitudinal part d_l of the photon Green's function to be constant we get for the Green's function $G(p)$ of the particle[13]

$$iG(p) = \frac{1-\gamma_5}{2}\frac{1}{\hat{p}} \exp\left(-\frac{e_{1\Lambda}^2}{4\pi} d_l \ln \frac{\Lambda^2}{p^2}\right). \tag{15}$$

Here $e_{1\Lambda}$ is the bare charge and Λ is the cut-off momentum. The vertex part, for $p_1 \sim p_2 \sim k \sim p$, has the form

$$\Gamma_n(p_1, p_2, k) = \gamma_n \frac{1-\gamma_5}{2} \exp\left(\frac{e_{1\Lambda}^2}{4\pi} d_t \ln \frac{\Lambda^2}{p^2}\right). \qquad (16)$$

Finally, for the transverse part d_t of the photon Green's function we get

$$d_{t\Lambda}(k^2) = \left(1 + \frac{e_{1\Lambda}^2}{6\pi} \ln \frac{\Lambda^2}{k^2}\right)^{-1}. \qquad (17)$$

The task of the renormalization program consists in eliminating Λ from Eq. (17), since the renormalization constants in G and Γ cancel each other. The function $d_{t\Lambda}(k^2)$ is given in conventional electrodynamics by

$$d_{t\Lambda}(k^2)_{\text{el}} = \left(1 + \frac{e_\Lambda^2}{3\pi} \ln \frac{\Lambda^2}{k^2 + 4m^2}\right)^{-1}. \qquad (18)$$

Here the mass is introduced in a manner described by Bogolyubov, Logunov and Shirkov.[14] In the limit $k^2 \to 0$ the value of $e_\Lambda^2 d_{t\Lambda}(k^2)_{\text{el}}$ equals the renormalized charge e^2, and it is natural to reexpress this quantity in terms of the renormalized charge:

$$e_\Lambda^2 d_{t\Lambda}(k^2)_{\text{el}} = e^2\left(1 - \frac{e^2}{3\pi} \ln \frac{k^2 + 4m^2}{4m^2}\right)^{-1},$$

$$e^2 = e_\Lambda^2\left(1 + \frac{e_\Lambda^2}{3\pi} \ln \frac{\Lambda^2}{m^2}\right)^{-1} \qquad (19)$$

In the case of Eq. (17), however, the point $k^2 = 0$ cannot be used for normalization since the function (17) vanishes there. It is therefore necessary to express the function $e_{1\Lambda}^2 d_{t\Lambda}(k^2)$ in terms of its value e_1^2 at some other point $k^2 = k_0^2$:

$$e_{1\Lambda}^2 d_{t\Lambda}(k^2) = e_1^2\left(1 - \frac{e_1^2}{6\pi} \ln \frac{k^2}{k_0^2}\right)^{-1},$$

$$e_1^2 = e_{1\Lambda}^2\left(1 + \frac{e_{1\Lambda}^2}{6\pi} \ln \frac{\Lambda^2}{k_0^2}\right)^{-1}. \qquad (20)$$

The renormalized function $d_t(k^2)$ is given by

$$d_t(k^2) = \left(1 - \frac{e_1^2}{6\pi} \ln \frac{k^2}{k_0^2}\right)^{-1}. \qquad (21)$$

The renormalization invariant quantity $6\pi e_1^{-2} - \ln k_0^2 = 6\pi e_{1\Lambda}^{-2} - \ln \Lambda t$ represents the only (dimensional) constant of the theory. We shall assume that $e_1^2 \ll 1$ for values of k_0 of the order of magnitude of energies likely to appear in experiments, and that therefore perturbation theory is applicable and d_t may be replaced by unity. The behavior of $d_t(k^2)$ for "superlarge" $k^2 \sim k_0^2 \exp(6\pi e_1^{-2})$ will not be considered here, since the difficulties that appear in this region should be removed by the same methods as are used in conventional electrodynamics.

Our results are easily generalized to the case of several charges. For example in the case of two bare charges e_Λ and $e_{1\Lambda}$, with e_Λ corresponding to a particle of mass m, we have instead of Eq. (18) the following expression for $d_{t\Lambda}$

$$d_{t\Lambda}(k^2) = \left(1 + \frac{e_\Lambda^2}{3\pi} \ln \frac{\Lambda^2}{k^2 + 4m^2} + \frac{e_{1\Lambda}^2}{6\pi} \ln \frac{\Lambda_1^2}{k^2}\right)^{-1}. \qquad (22)$$

One then eliminates e_Λ, Λ, $e_{1\Lambda}$ and Λ_1 by expressing d_t in terms of the two experimental values $e_0^2 = e_\Lambda^2 d_{t\Lambda}(k_0^2)$ and $e_1^2 = e_{1\Lambda}^2 d_{t\Lambda}(k_1^2)$. For example, for $k_1^2 = k_0^2$ we obtain for d_t instead of Eq. (21)

$$d_t(k^2) = \left(1 + \frac{e^2}{3\pi} \ln \frac{k_0^2 + 4m^2}{k^2 + 4m^2} + \frac{e_1^2}{6\pi} \ln \frac{k_0^2}{k^2}\right)^{-1}. \qquad (23)$$

For brevity we use below Eq. (21).

6. VACUUM POLARIZATION, SPECIFIC PROCESSES

For not ultrasmall distances $r \gg k_0^{-1} \exp(6\pi e_1^{-2})$ the field due to a point charge Ze is of the form

$$\varphi(r) = \frac{Ze}{2\pi^2} \int k^{-2} \left(1 + \frac{e_1^2}{6\pi} \ln \frac{k_0^2}{k^2}\right)^{-1} e^{i\mathbf{kr}} d\mathbf{k}$$

$$\approx \frac{Ze}{r}\left[1 - \frac{e_1^2}{3\pi}(\ln k_0 r + \text{const})\right] \qquad (24)$$

in correspondence with the usual expression[15] for $rm \ll 1$. In the general case of an external field $\sim V$, whose extent in space is $\sim a$, the induced charge density ρ_1 is of the order of magnitude $\rho_1 \sim -e_1^2 V a^{-2}$.

Equation (21) for d_t represents in the case of vacuum polarization the first term in an expansion in the external field. The field produces a pair and thereafter the produced particles are considered to be free and described by plane waves. Such an approximation is legitimate only for virtual particles with energies $\epsilon \gg e_1 V$. Since for electrons $\epsilon > 2m$ this condition is in their case usually fulfilled; in our case it is violated for momenta $p \lesssim e_1 V$. For such particles the external field is strong and they are produced in vacuum with a probability ~ 1. The resultant density ρ_2 may be estimated by filling a Fermi sphere with limiting momentum $p_0 \sim e_1 V$, so that $\rho_2 \sim -e_1(e_1 V)^3$. For $e_1^2(Va)^2 \ll 1$ the quantity ρ_2 may be ignored in comparison with ρ_1.

In the case of the Coulomb field $V \approx Zea^{-1}$ and the above condition reduces to $Zee_1 \ll 1$, i.e., the usual condition for applicability of perturbation theory. However for macroscopic systems ρ_2

may exceed ρ_1. In that case the screening charge q is $\rho_2 a^3$ and this screening will be small only if the condition

$$q/aV \sim e_1^4 (Va)^2 \underset{=}{\ll} 1 \qquad (25)$$

is satisfied.

The formulas for specific processes may be obtained from conventional electrodynamics by replacing m by 0 and by an appropriate selection of polarization. We remark that the cross sections for bremsstrahlung and pair production in the field of an atom will now be proportional to the atomic cross section a_0^2 instead of the usual m^{-2} obtained for electrons. The logarithmic angular divergence will be, in the absence of other causes, cut off by multiple scattering. In the high energy region $\epsilon > (Z^2 e^2 e_1^2 n a_0^4)^{-1}$ (where n is the density) multiple scattering becomes the determining factor and the conventional formulas for radiation and pair production in a condensed medium hold.[16,17]

7. EXPERIMENTAL ESTIMATE OF e_1

The severest restrictions on the size of e_1 apparently come from macroscopic physics. The absence of a mass makes the pair production process thresholdless and therefore electromagnetic radiation will produce pairs in any external field. For small values of e_1 simple pair production in the Earth magnetic field is likely to be the most intense process. If the Larmor radius of the resultant particles is less than the radius of the Earth, the particles will be trapped by the Earth magnetic field. As a result they will accumulate up to a density when annihilation becomes probable. If there is time for equilibrium to be established the particle density in a unit of volume will be of the order of photon density. Thus the Earth and the adjacent magnetic field region represent a plasma of such particles. The presence of this plasma should be detectable by various macroscopic effects.

It is likely that the strongest inequality on e_1 is obtained from considerations on the propagation of long radio waves. In a magnetized plasma electromagnetic waves are attenuated. The absence of this effect leads to a gross estimate

$$e_1 \lesssim 10^{-8} e, \qquad (26)$$

accurate to within one or two orders of magnitude.

Most likely the estimate (26) means that $e_1 = 0$; one might, however, ask what would be the consequences of the existence of a weak charge e_1 for the known neutrinos. For the β-decay neutrino such a possibility is unlikely. If it is assumed that e_1 is conserved then it follows from the experiments on measurements of the charge of the neutron and neutral molecules[18] that $e_1 \lesssim 10^{-15}$ e. Nonconservation of e_1 on the other hand, although it could be the reason for the weakness of the interaction, just like strangeness nonconservation is the reason for the weakness of hyperon decay interactions, leads to difficulties in connection with gauge invariance.

If, however, the two-neutrino hypothesis[19] is accepted then the introduction of a weak charge e_1 for the muon neutrino is not impossible. Assuming that e_1 is conserved we conclude that the charge of the muon differs from that of the electron by the amount e_1. This also explains the absence of the decays $\mu \rightarrow e + \gamma$, $\mu \rightarrow 3e$ to any order in the interaction,[20] and lepton conservation in the reactions $\pi \rightarrow \mu$, $K \rightarrow \mu$, $\mu \rightarrow e$ reduces simply to charge conservation.

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CALCULATION OF PROBABILITIES OF M3 TRANSITIONS AND SECOND-FORBIDDEN BETA TRANSITIONS IN THE NILSSON MODEL

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Expressions are obtained for the probabilities of M3 transitions and for second-forbidden beta transitions of pure Gamow-Teller type. The specific examination of the M3 transitions which are known in deformed nuclei indicates that the use of the corrected Nilsson representation is not necessary in these cases. A study of low-probability M3 transitions and second-forbidden beta transitions shows however that sometimes, when certain definite selection rules are satisfied, the use of the corrected Nilsson representation increases the transition probability by approximately a factor of two.

1. INTRODUCTION

EARLIER[1] we have calculated nuclear quadrupole moments on the assumption that they are caused by the Z protons moving independently in the deformed nuclear field which carries out a slow rotation which does not affect the individual proton motions. Such a model, which is analogous to that used by Inglis for calculating moments of inertia, gives the following quadrupole moment operator:*

$$\mathfrak{M}''_M(E2) = e\sum_{p=1}^{Z} r''^2_p Y_{2,M}(\vartheta''_p,\varphi''_p) = \sqrt{5/16\pi}\, eQ_M\ . \qquad (1)$$

Using the eigenfunctions of the Nilsson single-particle Hamiltonian, we showed that the use of the corrected representation, which takes account of the interaction of states with different quantum numbers N, is necessary for nuclei in the range $150 < A < 188$. In a succeeding paper[2] we came to the conclusion that it is not necessary to use the corrected representation for computing the probability of E2 gamma transitions, i.e, for transitions which are accompanied by a change in the internal structure of the nucleus. The reason for this is the different way in which the operators $r'^2 Y_{2,0}(\vartheta', \varphi')$ and $r'^2 T_{2,\nu}(\vartheta', \varphi')$ (with $\nu \neq 0$) transform when we go over to dimensionless coordinates.

The presence of a term containing only ρ^2 in the expression for the operator when $\nu = 0$ has the consequence that in the expression for the nuclear quadrupole moment, the correction term (which is proportional to a parameter ϵ, related to the deformation parameter δ) has the same order of magnitude as the main term, and consequently it becomes necessary to use the corrected representation.

Let us set $Q_0 = \langle Q_{M=0} \rangle$ and let us denote by $Q_0(\epsilon)$ and $Q_0(\delta)$ the theoretical values calculated for the operator (1) using, respectively, the improved and the original Nilsson representations, and let us denote by $Q_{0h}(\epsilon)$ and $Q_{0h}(\delta)$ the same values obtained by using the operator in the hydrodynamic approximation,

$$\mathfrak{M}''_M(E2) = e\sum_{p} r''^2_p Y_{2,M}(\vartheta''_p, \varphi''_p) + \frac{3}{4\pi} ZeR_0^2 \alpha^*_{2,M} \qquad (2)$$

(where R_0 is the nuclear radius). In addition, we denote by Q_{0N} the values calculated using Nilsson's formula:[3]

$$Q_{0N} \approx \tfrac{4}{5} ZR_0^2 \varepsilon \left(1 + \tfrac{1}{2}\varepsilon + \ldots\right),$$

which is derived from the Thomas-Fermi model on the assumption that $\langle r'^2 \rangle = \frac{3}{5} R_0^2$ for all the protons. The results obtained by us can then be represented by the table below (where the values of Q_0 are given in units of 10^{-24} cm^2). The experimental values of intrinsic quadrupole moments are taken from the work of Alder, A. Bohr et al.[4] The corresponding errors of the measurements vary between 10 and 20%.

We note that the values of $Q_0(\epsilon)$ are in good agreement with the experimental data, except for Lu^{175} and Hf^{177}, for which the discrepancy is somewhat greater. As expected, in both representations the hydrodynamic values are larger than the experimental values.

From a comparison of the values of $Q_0(\epsilon)$ and $Q_0(\delta)$, we see that the use of the corrected repre-

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*We use x'', y'', z'' for the space-fixed coordinate system, and x', y', z' for the coordinate system fixed on the nucleus.

Nucleus	Eu^{153}	Gd^{155}	Gd^{157}	Tb^{159}	Ho^{165}	Lu^{175}	Hf^{177}	Hf^{179}
ε_e	0.33	0.31	0.31	0.34	0.30	0.26	0.25	0.24
$Q_{0\,\mathrm{exp}}$, 10^{-24} cm^2	7.7	8.0	7.7	6.9	7.8	8.2	7.5	7.0
$Q_0(\varepsilon)$	7.3	7.5	7.5	7.6	7.1	6.6	6.5	6.4
Q_{0N}	7.9	7.6	7.7	8.8	8.0	7.5	7.3	7.1
$Q_0(\delta)$	3.7	3.9	3.9	4.0	3.5	3.2	3.7	3.3
$Q_{0h}(\varepsilon)$	12	11.9	12	12.5	11.4	11.5	11.2	10.5
$Q_{0h}(\delta)$	10.2	10.1	10.2	10.3	9.6	9.5	9.5	9.3

sentation is necessary for computing nuclear quadrupole moments using the operator (1) ($Q_0(\epsilon) \approx 2Q_0(\delta)$). In the case where one applies operator (2), the difference which appears when one uses the two representations is much smaller, because the collective term does not change when we go over from one representation to the other.

In the present paper we consider other nuclear phenomena in whose mathematical description the operator $r'^2 Y_{2,0}(\vartheta', \varphi')$ appears, and where the use of the corrected representation may be necessary. Phenomena of this type are the M3 gamma ray transitions and second-forbidden beta transitions.

2. M3 TRANSITIONS

The reduced probability for an M3 transition is given by the expression

$$B(M3;\ J_i \to J_f) = \sum_{M,\,M_f} |\langle \Omega_f;\ J_f K_f M_f | \mathfrak{M}_M(M3) | \Omega_i;\ J_i K_i M_i \rangle|^2. \qquad (3)$$

In the space-fixed coordinate system x'', y'', z'', we write the transition operator in the form

$$\mathfrak{M}_M(M3) = \frac{e\hbar}{2Mc} \sum_p \left(g_s \mathbf{s} + \frac{1}{2} g_l \mathbf{l} \right) \nabla_p [r_p''^3 Y_{3,M}(\vartheta_i'', \varphi_i'')]. \qquad (4)$$

In the coordinate system fixed in the nucleus (x', y', z'), we consider the expression

$$\mathbf{l}(\nabla r'^3 Y_{3,\nu}) = \sqrt{7/5}\,[l_z r'^2 \sqrt{9-\nu^2}\, Y_{2,\nu}$$
$$+ \tfrac{1}{2}\sqrt{(3-\nu)(2-\nu)}\, l_- r'^2 Y_{2,\nu+1}$$
$$- \tfrac{1}{2}\sqrt{(3+\nu)(2+\nu)}\, l_+ r'^2 Y_{2,\nu-1}]$$

(where $l_\pm = l_x \pm i l_y$) and an analogous expression for the operator $\mathbf{s}\,(\nabla r'^3 Y_{3,\nu})$. Transforming to the new variables

$$\xi = x' \sqrt{M\omega_x/\hbar}, \qquad \eta = y' \sqrt{M\omega_y/\hbar},$$
$$\zeta = z' \sqrt{M\omega_z/\hbar}, \qquad (5)$$

where

$$\omega_x = \omega_y = \omega_0(\varepsilon)\left(1 + \tfrac{1}{3}\varepsilon\right), \quad \omega_z = \omega_0(\varepsilon)\left(1 - \tfrac{2}{3}\varepsilon\right),$$
$$\omega_0(\varepsilon) \approx \omega_0^0\left(1 + \tfrac{1}{9}\varepsilon^2\right),$$

and using the same nuclear wave function as previously,[1,2] we obtain, in place of (3), the formula*

$$B(M3;\ J \to J_f) = \left(\frac{e\hbar}{2Mc}\right)^2 \left(\frac{\hbar}{M\omega_0(\varepsilon)}\right)^2 \frac{7}{16\pi} \langle J_i 3 K_i k | J_f K_f \rangle^2 G_{M3}^2(\varepsilon), \qquad (6)$$

$$k = K_f - K_i, \qquad G_{M3}(\varepsilon) \approx G_{M3}^t + \varepsilon G_{M3}^{t'}. \qquad (7)$$

G_{M3}^t in Eq. (7) coincides with the term $G_{M3}(\delta)$ found by Nilsson[3] (his formula 36c), with the one difference that the quantum numbers N, l and Λ should be replaced by N_t, l_t and Λ_t, determined in the corrected representation.†

We found the following expression for the correction term $\epsilon G_{M3}^{t'}$:

$$G_{M3}^{t'} = \sum_{l_f,\,l_i} \langle N_f l_f | \rho^2 | N_i l_i \rangle \sum_{\Lambda_f,\,\Lambda_i,\,\Sigma_f,\,\Sigma_i} {}_{l_f,\,\Lambda_f}a_{l_i,\,\Lambda_i}$$
$$\times \Big\{ g_s \Big[\langle l_i 200 | l_f 0 \rangle \sqrt{\frac{2l_i+1}{2l_f+1}} \big(A(k)\, \delta_{\Sigma_f,\,\Sigma_i}$$
$$\times (-1)^{\Sigma_i - 1/2} u(k) \langle l_i 2 \Lambda_i k | l_f \Lambda_f \rangle$$
$$+ B(k)\, \delta_{\Sigma_f,\,-1/2} \delta_{\Sigma_i,\,1/2} v(k) \langle l_i 2 \Lambda_i k + 1 | l_f \Lambda_f \rangle$$
$$- C(k)\, \delta_{\Sigma_f,\,1/2} \delta_{\Sigma_i,\,-1/2} w(k) \langle l_i 2 \Lambda_i k - 1 | l_f \Lambda_f \rangle \big)$$
$$+ \tfrac{1}{3}\left(1 + \tfrac{1}{3}\varepsilon\right) \sqrt{5/4\pi} \big(A(k)\, \delta_{k,0} \delta_{\Sigma_f,\,\Sigma_i} (-1)^{\Sigma_i - 1/2}$$
$$+ B(k)\, \delta_{k,\,-1} \delta_{\Sigma_f,\,-1/2} \delta_{\Sigma_i,\,1/2}$$
$$- C(k)\, \delta_{k,\,1} \delta_{\Sigma_f,\,1/2} \delta_{\Sigma_i,\,-1/2} \big) \delta_{l_i l_f} \delta_{\Lambda_i,\,\Lambda_f} \Big]$$
$$+ \tfrac{1}{2} g_l \delta_{\Sigma_f,\,\Sigma_i} \Big[\langle l_i 200 | l_f 0 \rangle \sqrt{\frac{2l_i+1}{2l_f+1}} \big[A(k)\, 2\Lambda_f u(k)$$
$$\times \langle l_i 2 \Lambda_i k | l_f \Lambda_f \rangle + B(k) \big(\sqrt{(l_f - \Lambda_f)(l_f + \Lambda_f + 1)}\, v(k)$$
$$\times \langle l_i 2 \Lambda_i k + 1 | l_f \Lambda_f + 1 \rangle + p(k) \langle l_i 2 \Lambda_i k | l_f \Lambda_f \rangle \big)$$
$$- C(k) \big(\sqrt{(l_f + \Lambda_f)(l_f - \Lambda_f + 1)}\, w(k)$$
$$\times \langle l_i 2 \Lambda_i k - 1 | l_f \Lambda_f - 1 \rangle + q(k) \langle l_i 2 \Lambda_i k | l_f \Lambda_f \rangle \big) \big]$$
$$+ \tfrac{1}{3} \sqrt{5/4\pi} \left[\left(1 + \tfrac{1}{3}\varepsilon\right) A(k) \cdot 2\Lambda_f \delta_{k,\,0} \delta_{l_f,\,l_i} \delta_{\Lambda_f,\,\Lambda_i} \right.$$

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*The details of the calculation will be published elsewhere.[5]

†For simplicity, in the following we omit the subscript t in writing the quantum numbers.

$$+ B(k)\left(\left(1+\tfrac{1}{3}\varepsilon\right)\sqrt{(l_f-\Lambda_f)(l_f+\Lambda_f+1)}\right.$$
$$\times\,\delta_{k,-1}\delta_{l_f,l_i}\delta_{\Lambda_i,\Lambda_f+1}+\sqrt{3}\left(2+\tfrac{1}{3}\varepsilon\right)\delta_{k,0}\delta_{l_f,l_i}\delta_{\Lambda_i;\Lambda_f}\Big)$$
$$- C(k)\left(\left(1+\tfrac{1}{3}\varepsilon\right)\sqrt{(l_f+\Lambda_f)(l_f-\Lambda_f+1)}\right.$$
$$\times\,\delta_{k,1}\delta_{l_f,l_i}\delta_{\Lambda_i,\Lambda_{f-1}}+\sqrt{3}\left(2+\tfrac{1}{3}\varepsilon\right)\delta_{k,0}\delta_{l_f,l_i}\delta_{\Lambda_i,\Lambda_f}\Big)\Big]\Big]\Big\}. \tag{7a}$$

Here

$$\rho^2=\xi^2+\eta^2+\zeta^2; \qquad A(k)=\sqrt{9-k^2},$$

$$B(k)=\sqrt{(3-k)(2-k)}, \qquad C(k)=\sqrt{(3+k)(2+k)},$$

$$p(k)=\begin{cases}1 & \text{for } k=1\\ -1 & \text{for } k=-2\\ \sqrt{1/6} & \text{for } k=0,\\ -\sqrt{1/6} & \text{for } k=-1\end{cases} \qquad q(k)=\begin{cases}1 & \text{for } k=-1\\ -1 & \text{for } k=2\\ \sqrt{1/6} & \text{for } k=0\\ -\sqrt{1/6} & \text{for } k=1\end{cases},$$

$$u(k)=\begin{cases}0 & \text{for } k=\pm 3\\ -1/3 & \text{for } k=\pm 2\\ 1/6 & \text{for } k=\pm 1\\ 1/3 & \text{for } k=0\end{cases}, \qquad v(k)=\begin{cases}-1/3 & \text{for } k=-3.1\\ 1/6 & \text{for } k=-2.0\\ 1/3 & \text{for } k=-1\\ 0 & \text{for } k=-2,3\end{cases},$$

$$w(k)=\begin{cases}0 & \text{for } k=-3,\,-2\\ -1/3 & \text{for } k=-1.3\\ 1/6 & \text{for } k=0.2\\ 1/3 & \text{for } k=1\end{cases}.$$

3. SECOND-FORBIDDEN BETA TRANSITIONS

The set of nuclear matrix elements which appear in the form factors of second-forbidden β spectra are written, following Konopinski,[6] as follows:*

$$S, V:\ \int(\beta)\,\mathcal{Y}_{2,M}(\mathbf{r}), \quad \Delta J=2;$$

$$T, A:\ \int(\beta)\,\mathcal{Y}_{2,M}([\boldsymbol{\sigma}\mathbf{r}]), \quad \Delta J=2;$$

$$T, V:\ \int(\beta)\,\mathcal{Y}_{2,M}(\boldsymbol{\alpha}), \quad \Delta J=2;$$

$$T, A:\ \int(\beta)\,\mathcal{Y}_{3,M}(\boldsymbol{\sigma}), \quad \Delta J=3,$$

where, in the space-fixed coordinate system,

$$\mathcal{Y}_{2,M}(\mathbf{r})=r''^2 Y_{2,M}(\vartheta'',\varphi''),$$

$$\mathcal{Y}_{3,M}(\boldsymbol{\sigma})=\sqrt{7/5}\Big[\sqrt{9-M^2}\,r''^2 Y_{2,M}s_z$$
$$+\tfrac{1}{2}\sqrt{(3-M)(3-M-1)}\,r''^2 Y_{2,M+1}s$$
$$-\tfrac{1}{2}\sqrt{(3+M)(3+M-1)}\,r''^2 Y_{2,M\ 1}s_+\Big],$$

$$\mathcal{Y}_{2,M}(\boldsymbol{\alpha})=\tfrac{1}{2}r''\sqrt{5/3}\Big[\sqrt{4-M^2}\,Y_{1,M}\alpha_z$$
$$-\sqrt{(2+M)(2+M-1)}\,Y_{1,M-1}\alpha_+$$
$$+\sqrt{(2-M)(2-M+1)}\,Y_{1,M+1}\alpha_-\Big],$$

$$\mathcal{Y}_{2,M}([\boldsymbol{\sigma}\mathbf{r}])=\frac{i}{2\hbar}\boldsymbol{\sigma}\mathbf{L}\mathcal{Y}_{2,M}(\mathbf{r})$$
$$=\frac{i}{2\hbar^2}(j^2-L^2-s^2)\,r''^2Y_{2,M}(\vartheta'',\varphi'').$$

The matrix element containing the operator $\mathcal{Y}_{3,M}(\sigma)$ has a form which is very similar to the matrix element $\mathfrak{M}_M(M3)$, which corresponds to M3 gamma radiation. The β transitions described by this matrix element, the so-called "unique" transitions, have the remarkable property that their spectra are easily determined. The change in parity for such a transition is $\Delta\pi=(-1)^{\Delta J+1}$.

The corresponding values of $f_2\tau$ have the following form (where we use Nilsson's[3] notation; x is the fraction of pure Gamow-Teller interaction):

$$f_2\tau=B_g[xD_{G.T.}(2)]^{-1}, \qquad B_g=2\pi^3\hbar\ln 2/g^2m_e^5c^4, \tag{8}$$

where the reduced probability of transition is

$$D_{G.T.}(2)=S^2(2)\sum_{M,M_f}\left|\int\mathcal{Y}_{3,M}(\boldsymbol{\sigma})\right|^2,$$

$$\int\mathcal{Y}_{3,M}(\boldsymbol{\sigma})$$
$$=\langle\Omega_f;J_fK_fM_f\Big|\sum_p s_p\nabla_p[r_p''^3Y_{3,M}(\vartheta_p'',\varphi_p'')\tau_\pm^p]\Big|\Omega_i;\ J_iK_iM_i\rangle,$$

$$S(2)=\sqrt{\frac{4\pi 2^5}{7!}\frac{(3!)^2}{3}\left(\frac{m_ec}{\hbar}\right)^2}.$$

Proceeding in the same way as for M3 transitions, we get for $D_{G.T.}(2)$ the expression

$$D_{G.T.}(2)=S^2(2)\frac{7}{16\pi}\left(\frac{\hbar}{M\omega_0(\varepsilon)}\right)^2\langle J_i3K_ik|J_fK_f\rangle^2\gamma_3^2(\varepsilon). \tag{9}$$

Here $\gamma_3(\epsilon)=G_{M3}(\epsilon)$ $(g_s=1,\ g_l=0)$, where the expression for $G_{M3}(\epsilon)$ is defined by (7). The operators $\mathcal{Y}_{2,M}(\mathbf{r})$ and $\mathcal{Y}_{2,M}(\boldsymbol{\alpha}\times\mathbf{r})$ are proportional to the operator $\mathfrak{M}_M(E2)$ of radiation theory. For this reason the square moduli of the matrix elements appearing in the expressions for the quantity $f_2\tau$ are very similar to the expression for the reduced transition probability B(E2):

$$\sum_{M,M_f}\left|\int\mathcal{Y}_{2,M}(\mathbf{r})\right|^2=\frac{\hbar}{M\omega_0(\varepsilon)}\frac{5}{4\pi}\langle J_i2K_ik|J_fK_f\rangle^2\gamma_2^2(\varepsilon), \tag{10}$$

$$\gamma_2(\varepsilon)=G_{E2}^t+\varepsilon G_{E2}^{t'}. \tag{11}$$

In Eq. (11) G_{E2}^t is identical with the term found by Nilsson [formula (35c) in reference 3], where, however, the quantum numbers have the subscript t, and $G_{E2}^{t'}$ has the form

$$G_{E2}^{t'}=\sum_{l_i,l_f}\langle N_fl_f|\rho^2|N_il_i\rangle\sum_{\Lambda_f,\Lambda_i,\Sigma_i,\Sigma_f}a_{l_f,\Lambda_f}a_{l_i,\Lambda_i}\delta_{\Sigma_f,\Sigma_i}$$
$$\times\Big[\sqrt{(2l_i+1)/(2l_f+1)}\,\langle l_i200|l_f0\rangle\langle l_i2\Lambda_ik|f_f\Lambda_f\rangle p(k)$$
$$+\tfrac{1}{3}\left(1+\tfrac{1}{3}\varepsilon\right)\delta_{k,0}\delta_{l_i,l_f}\delta_{\Lambda_i,\Lambda_f}\Big],$$

where

$$p(k)=\begin{cases}1/3 & \text{for } k=0\\ 1/6 & \text{for } k=\pm 1\\ -1/3 & \text{for } k=\pm 2\end{cases}.$$

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*$[\boldsymbol{\sigma}\mathbf{r}]=\boldsymbol{\sigma}\times\mathbf{r}$.

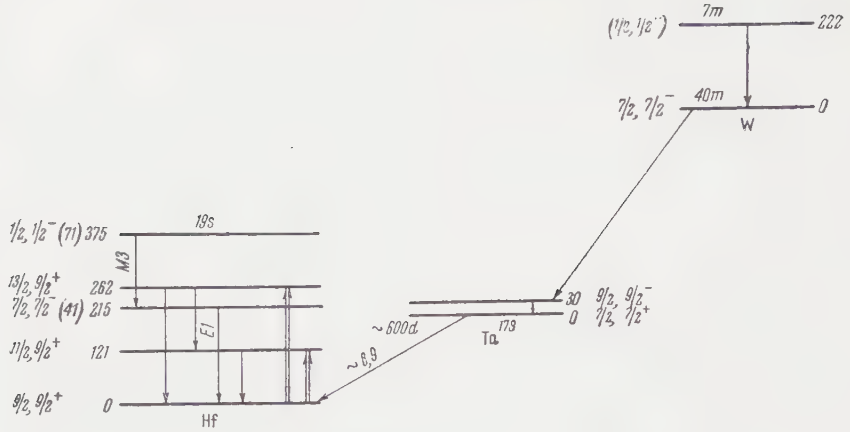

FIG. 1

4. SOME SPECIFIC CASES

One may anticipate that, just as in the calculation of quadrupole moments, the use of the corrected representation will be important in those cases where a contribution proportional to ρ^2 appears in the expression for the operator in the correction term.

For deformed nuclei, using the data of Mottelson and Nilsson,[7] we found three M3 transitions. These are shown in Figs. 1 and 2 (where we indicate for each level, in addition to its energy, the values of J, K and the parity).

In the case of Hf^{179} and W^{179}, the final state of the transition is the same (41 in Nilsson's notation[3]). In Hf^{179} the initial level is 71, whereas for W^{179} we have either 71 or 63. In both cases, $\eta \approx 4$,* and the corresponding transitions are isomeric: the half-life of the initial level corresponding to the first transition is 19 sec, while that of the initial level for the second transition is 7 min. For both transitions, k = 3, and consequently, according to the selection rule on k in Eq. (7a), the contribution of the term in ρ^2 from the expression for $\mathfrak{M}_M(M3)$ is zero. In the case of Hf^{179} we found $G^2_{M3}(\delta) = 0.553 \times 10^2$ and $G^2_{M3}(\epsilon) = 0.471 \times 10^2$. Consequently, as expected, in this case the use of the corrected representation does not essentially change the expression for the transition probability. The same conclusions apply to W^{179}.

For Ta^{181} (Fig. 2), although the levels between which the M3 transition occurs are different (43 and 25), nevertheless in this case also k = 3, and for $\eta = 4$ we found $G^2_{M3}(\delta) = 0.174 \times 10^3$ and $G^2_{M3}(\epsilon) = 0.161 \times 10^3$.

The theoretical half-life computed using the values $G^2_{M3}(\epsilon)$ is approximately 10 sec for the ($\frac{1}{2}$, $\frac{1}{2}^-$) state in Hf^{179}; for the ($\frac{1}{2}$, $\frac{1}{2}^+$) state in Ta^{181} it is 1.9×10^{-4} sec, when we include the E2 transition ($\frac{1}{2}$, $\frac{1}{2}^+$) → ($\frac{5}{2}$, $\frac{5}{2}^+$) which we calculated previously.[2] For the initial level of the M3 transition in W^{179} we found a value of τ of the same order as the corresponding quantity for the ($\frac{1}{2}$, $\frac{1}{2}^-$) state of Hf^{179}, i.e., a value which is an order of magnitude smaller than the experimental result. The M3 transition in Hf^{179} is strongly suppressed; the experimental value of the suppression factor is $F_{exp} \approx 10^{-3}$. The corresponding theoretical factor* was $F_{th} = 2 \times 10^{-2}$.

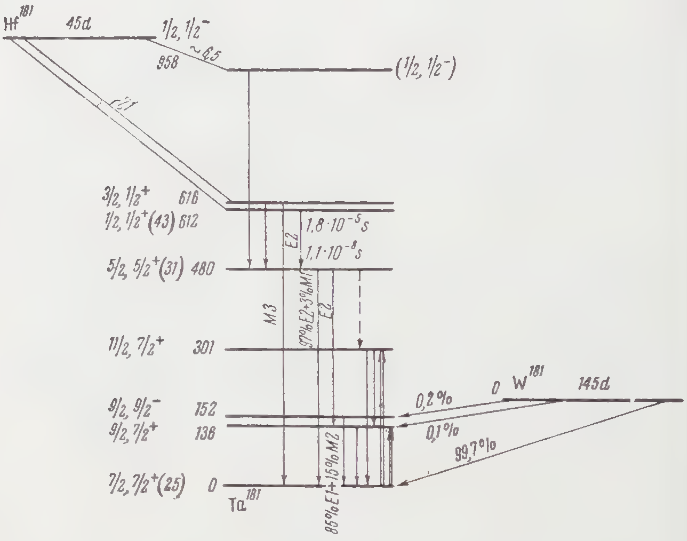

FIG. 2

In order to examine cases where, in the expression for $G_{M3}(\epsilon)$, the selection rules on k allow nonzero contributions of the term in ρ^2 (i.e., cases where k takes on one of the values −1, 0, or 1) we have considered transitions having low probability, which are as yet not observed, which according to the selection rules might be M3, and for which k takes on one of the values enumerated.

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*η denotes the parameter introduced by Nilsson,[3] which is related to the nuclear deformation δ'.

*Our factor F is equal to H^{-1}, where H is the factor used by Mottelson and Nilsson.[7]

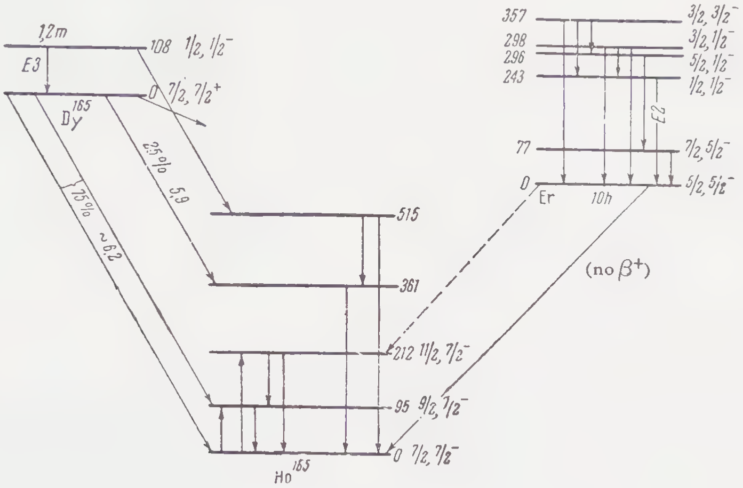

FIG. 3

In the case of Ta^{181}, a transition of this type is the one indicated by the dashed line in Fig. 2. However, it turned out in this case that when we substituted the values of the coefficients $a_{l,\Lambda}$, the term of G^t_{M3} of interest to us (which was different from zero since $k = 1$) was small compared with the others, and the use of the corrected representation gave nothing new.

Analogous results are obtained in the case of second forbidden β transitions. In deformed nuclei, not a single case of a second forbidden transition has yet been discovered, so we turned to the examination of doubtful cases. Among the nuclei with $A = 161$,[7] the transition $(3/2, 3/2^+)$ Tb → $(9/2, 5/2^+)$ Dy, with $k = 1$, might be a unique β^- transition. With $\eta = 6$, we found no significant difference between $\gamma_3^2(\delta)$ and $\gamma_3^2(\epsilon)$. However, for nuclei with $A = 165$ (cf. Fig. 3) the transition $(5/2, 5/2^-)$ Er → $(11/2, 7/2^-)$ Ho should be studied using the corrected representation, since for $\eta = 6$ we have $\gamma_3^2(\delta) = 0.13 \times 10^2$ and $\gamma_3^2(\epsilon) = 0.256 \times 10^2$. The result of this is that the difference resulting from using the two representations appears even in the log $f_2\tau$ values:

$$\log f_2\tau(\varepsilon) = 15.8, \quad \log f_2\tau(\delta) = 16.1.$$

Similarly, in the case of nuclei with $A = 177$, for the β^- transition $(7/2, 7/2^-)$ Lu → $(11/2, 7/2^-)$ Hf[7] the absolute squares of some of the matrix elements (of type 10) are twice as great for the corrected as for the original Nilsson representation. The reason is that for such a transition $k = 0$.

5. CONCLUSION

Unlike the case of the calculation of the quadrupole moments of nuclei in the region $150 < A < 188$, where the use of the corrected Nilsson representation was always necessary, for M3 gamma transitions and second forbidden β transitions the use of the corrected representation is limited to those cases where $k = K_f - K_i = -1$, 0, or 1, and where the coefficients $a_{l,\Lambda}$ of the wave functions corresponding to the initial and final states are sufficiently large. Such cases have not yet been discovered in deformed nuclei. The reason for the difference in the two calculations is the following. Because of the normalization condition $\Sigma a^2_{l,\Lambda} = 1$, the coefficients of the wave functions do not appear in the expression for the main correction term for the quadrupole moment; however, in our case the normalization condition cannot be used in the main correction term, since the coefficients $a_{l,\Lambda}$ refer to different states. Moreover, in the quadrupole moment calculations the condition $k = 0$ is always satisfied, which guarantees the appearance of the correction term in which we are interested.

In conclusion, I express my gratitude to Prof. S. Titeica for his interest in the work.

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QUANTUM KINETIC EQUATION IN THE PRESENCE OF MUTUAL ENTRAINMENT OF ELECTRONS AND PHONONS

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A theory is developed for the thermoelectric coefficients in a quantized magnetic field in the case of mutual entrainment of phonons and electrons. A set of kinetic equations for the phonon distribution functions, and the electron density (nondiagonal) matrix f is obtained with the aid of the diagram technique developed in reference 5. It is shown that solution of the integral equations for g and the diagonal part of f should precede the expansion in powers of $(\omega\tau)^{-1} \ll 1$ (ω is the Larmor frequency and τ the electron relaxation time). An arbitrary electron and phonon spectrum is assumed.

THE important role of the deviation of the phonon distribution function from equilibrium in thermoelectric phenomena, i.e., the effect of entrainment of the electrons by phonons (or vice-versa), was reported some time ago.[1,2] The starting point in these papers was the Boltzmann transport equation for the distribution functions of the phonons and electrons, with allowance for deviations of both systems from equilibrium. However, in the case when the energy is quantized and the distance between discrete levels is commensurate with or greater than $T = \beta^{-1}$ (T is the temperature in energy units), the problem is radically changed. Thus, for example, in a magnetic field the electron velocity component transverse to the field has no diagonal matrix elements, so that to calculate the corresponding current components it is necessary to know the nondiagonal elements of the density matrix, and the problem cannot even be formulated in terms of the customary classical theory.

The purpose of the present investigation was to obtain suitable quantum equations for the entrainment theory. Although we shall be interested in what follows in the thermoelectric tensor, we shall determine the Peltier tensor, which is related to the former by simple symmetry relations[3] (the Herring Π-approach[2]). In investigations of thermoelectric phenomena we deal with entrainment of phonons by electrons acted upon by an electric field. In this case the total heat flow consists of two parts, the ordinary part produced by the electrons, and a part due to the entrainment of the phonons by the electrons and produced by the phonons.

1. SYSTEM OF KINETIC EQUATIONS FOR THE ELECTRONS AND PHONONS IN A QUANTIZING MAGNETIC FIELD

If a crystal with arbitrary electron spectrum $\epsilon(\hat{P})$ is placed in a magnetic field $H = (0, 0, H)$, we can choose the energy operator in the form $\hat{\epsilon} = \hat{\epsilon}(P_x, (eH/c)(\hat{x}_0 - \hat{x}), \hat{P}_z)$, where $\hat{x}_0 = (c/eH)\hat{P}_y$. The corresponding wave functions are of the form

$$\psi = (L_y L_z)^{-1/2} \exp\,[(i\hbar)^{-1}(P_y y + P_z z)]\,\varphi_{nP_z}(x - x_0) \quad (1.1)$$

$L_{y,z}$ are the crystal dimensions in the y, z direction.

The functions φ satisfy the equation

$$\varepsilon(\hat{P}_x, -(eH/c)\,x,\ P_z)\,\varphi_{nP_z}(x) = \varepsilon_n(P_z)\,\varphi_{nP_z}(x),$$

where $\epsilon_n(P_z)$ are the eigenvalues of the energy $\hat{\epsilon}$.

If the electrons do not interact with one another, the complete Hamiltonian of the electron-phonon system is of the form

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{V}; \qquad \hat{\mathscr{H}}_0 = \sum_\alpha \hat{a}_\alpha^+ \hat{a}_\alpha \varepsilon_\alpha + \sum_{\mathbf{q}} \hat{b}_{\mathbf{q}}^+ \hat{b}_{\mathbf{q}} \hbar\omega_{\mathbf{q}},$$

$$\hat{V} = \sum_{\mathbf{q}} \sum_{\alpha\alpha'} \sum_j V_{ed}(\mathbf{q})\, J_{\alpha\alpha'}(\mathbf{q})\, \hat{a}_\alpha^+ \hat{a}_{\alpha'} \exp\,[-i\mathbf{q}\mathbf{r}_j/\hbar]$$
$$+ \sum_{\mathbf{q}} \sum_{\alpha\alpha'} \{c_{\mathbf{q}} \hat{b}_{\mathbf{q}} J_{\alpha\alpha'}(\mathbf{q}) + c_{\mathbf{q}}^* b_{\mathbf{q}}^+ J_{\alpha'\alpha}^*(\mathbf{q})\}\, \hat{a}_\alpha^+ \hat{a}_{\alpha'} + \hat{V}_{ff} + \hat{V}_{fd}.$$

Here α is the aggregate of the quantum numbers of the electron (nP_yP_z), $\omega_{\mathbf{q}}$ is the cyclic frequency of a phonon with momentum $\mathbf{q}$; $V_{ed}(\mathbf{q})$ is the Fourier component of the potential of the interaction between the electrons and the defects; $\mathbf{r}_j$ is the coordinate of the j-th defect (the eventual averaging is over $\mathbf{r}_j$); $J_{\alpha\alpha'}(\mathbf{q})$ is the matrix element of the operator $\exp[i\mathbf{q}\cdot\mathbf{r}/\hbar]$; $c_{\mathbf{q}}$ characterizes the interaction between the electrons and the phonons, and is proportional to $q^{1/2}$ for small q; $\hat V_{ff}$ and $\hat V_{fd}$ are the operators of the phonon-phonon and phonon-defect interactions.

In the presence of a constant and homogeneous electric field **E**, the deviation of the density matrix $\hat\rho_1$ of the system from the equilibrium matrix $\hat\rho_0$ is[4, 5]

$$\hat\rho_1 = \hat\rho_0 \int_{-\infty}^{0} d\tau e^{s\tau} \int_0^\beta d\lambda \int d^3 r e \hat{\mathbf J} (\mathbf r, \tau - i\hbar\lambda)\, \mathbf E$$

$$= \hat\rho_0 \int_{-\infty}^{0} d\tau e^{s\tau} \int_0^\beta d\lambda e \hat{\mathbf v} (\tau - i\hbar\lambda)\, \mathbf E, \quad s \to +0,$$

where $\hat{\mathbf v}$ and $\hat{\mathbf J}(\mathbf r)$ are the electron-velocity and the particle-flux-density operators.

We introduce, following Konstantinov and Perel',[4] the single-particle matrices

$$f_{\beta'\beta} = \operatorname{Sp} \hat\rho_1 \hat a_\beta^+ a_{\beta'} = e\mathbf E \sum_{\alpha\alpha'} (\mathbf v)_{\alpha\alpha'} \int_{-\infty}^{0} d\tau e^{s\tau} \int_0^\beta d\lambda$$

$$\times \operatorname{Sp}\left\{\hat\rho_0 T_C \left\{\exp\left[(i\hbar)^{-1}\int_C \hat V(z)\, dz\right] (\hat a_\beta^+ \hat a_{\beta'})_{-i\hbar\lambda} (\hat a_\alpha^+ \hat a_{\alpha'})_\tau\right\}\right\},$$

$$g_{\mathbf q'\mathbf q} = \operatorname{Sp} \hat\rho_1 \hat b_{\mathbf q}^+ \hat b_{\mathbf q'} = e\mathbf E \sum_{\alpha\alpha'} (\mathbf v)_{\alpha\alpha'} \int_{-\infty}^{0} d\tau e^{s\tau} \int_0^\beta d\lambda$$

$$\times \operatorname{Sp}\left\{\hat\rho_0 T_C \left\{\exp\left[(i\hbar)^{-1}\int_C \hat V(z)\, dz\right] (\hat b_{\mathbf q}^+ \hat b_{\mathbf q'})_{-i\hbar\lambda} (\hat a_\alpha^+ \hat a_{\alpha'})_\tau\right\}\right\}. \tag{1.2}$$

The integration contour C is shown in Fig. 1 (from τ via $\tau - i\hbar\lambda$ to $-i\hbar\lambda$, again to $\tau - i\hbar\lambda$, and finally to $\tau - i\hbar\beta$). The symbol T_C denotes the ordering of the Heisenberg operators along the contour C.

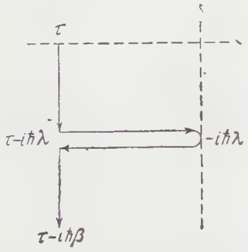

FIG. 1. The contour of integration C.

Konstantinov and Perel'[4] have obtained an equation which the matrix f satisfies when the phonons are not entrained by electrons (g = 0). Using their method, we readily obtain a system of transport equations for the matrices f and g. The general form of this system is most conveniently formulated by using the diagram technique which they introduced, and which we must generalize somewhat because the eigenfunctions of the operator $\mathcal{H}_0$ are not plane waves.

We shall represent the vertex corresponding to the electron-phonon interaction and containing $\hat a_\alpha^+ \hat a_{\alpha'} \hat b_{\mathbf q}$ (or $\hat a_\alpha^+ \hat a_{\alpha'} \hat b_{\mathbf q}^+$) by means of a point, on the contour C, at which three lines meet: the incoming (α') and the outgoing (α) electron lines (solid) and one incoming ($\hat b_{\mathbf q}$) or outgoing ($\hat b_{\mathbf q}^+$) phonon line (dotted). The rules for calculating the diagrams are as follows:

1) To each "regular" electron line (i.e., moving from the earlier to the later point along the contour) there corresponds a factor $(1 - n_\alpha)$, while to each "irregular" line (moving in the opposite direction) corresponds a factor n_α. To each "regular" phonon line corresponds a factor $1 + N_{\mathbf q}$, and to each "irregular" one — a factor $N_{\mathbf q}$ (n_α and $N_{\mathbf q}$ are the Fermi and Planck functions, respectively).

2) To each vertex corresponds a factor $c_{\mathbf q} J_{\alpha\alpha'}(\mathbf q)$, if the incoming lines are α' and $\mathbf q$ and the outgoing line is α, and a factor $c_{\mathbf q}^* J_{\alpha\alpha'}^*$ if the outgoing line is $\mathbf q$. In interactions with defects, incoming "defect" lines with no directions appear at the points. The corresponding factor is

$$V_{ed}(\mathbf q)\, J_{\alpha\alpha'}(\mathbf q) \exp\left[-i\mathbf q \mathbf r_j/\hbar\right].$$

3) To each point corresponds a factor $(i\hbar)^{-1}$ on the upper horizontal part of the contour C, a factor $(-i\hbar)^{-1}$ on the lower part, and a factor -1 on the vertical part.

4) Each diagram is multiplied by $(-1)^M$, where M is the number of intersections of the electron lines with each other. The electron lines approach the points only in a manner such that their direction coincides with the direction of motion along the contour near this point (see, for example, Figs. 8b, e, h, below).

5) To each interval between the points nearest in time on a horizontal part (upper or lower) corresponds a factor obtained in the following manner. The interval is cut by a vertical line and the factor $(s + i\omega_{MN})^{-1}$ is set up, where $\omega_{MN} = \hbar^{-1}(E_M - E_N)$, and E_M (or E_N) is the sum of the energies of the lines that cross the cutting line from left to right (or from right to left). If some cutting line crosses only two electron (or two phonon) lines, then the corresponding factor has a value s^{-1} when $\alpha = \alpha'$ (or $q = q'$) and diverges as $s \to 0$. This is the basis of the separation of the irreducible parts of the diagrams (see below).

6) A Laplace transform is taken in the variable $\beta = T^{-1}$ (T is the temperature in energy units):

$$f_{\alpha\alpha'}(\sigma) = \int_0^\infty d\beta e^{-\beta\sigma} f_{\alpha\alpha'}(\beta)$$

and analogously for $g_{qq'}$. The quantity β, which enters in $\hat\rho_0$, is not regarded as an integration variable. Then to each interval between two nearest points on the vertical part corresponds a factor $(\sigma + \hbar\omega_{MN})^{-1}$, where E_M (or E_N) is the energy of the lines that cross the horizontal cutting line from top to bottom (or from bottom to top).

In setting up all the possible sections, account is taken also of intervals whose boundary points are τ, $\tau - i\hbar\lambda$, $-i\hbar\lambda$, and $\tau - i\hbar\beta$.

The system of transport equations for f and g can be presented graphically by means of this technique as follows:

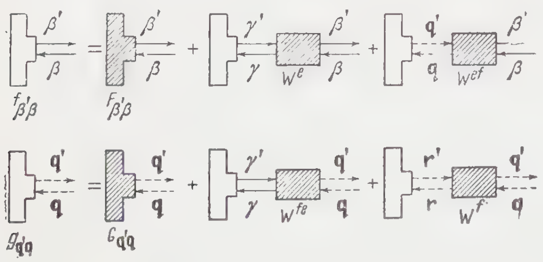

FIG. 2

The unshaded figures with two solid (or dotted) lines $\beta\beta'$ (or $\mathbf{q}'\mathbf{q}$) on the right represent the matrices $f_{\beta'\beta}$ (or $g_{\mathbf{q}'\mathbf{q}}$), which are equal to the sum of all the diagrams with all possible numbers and placements of the points on the horizontal and vertical sections, and with two "free" electron ($\beta'\beta$) (or phonon, $\mathbf{q}'\mathbf{q}$) lines on the right. The shaded figures $F_{\beta'\beta}$ (or $G_{\mathbf{q}'\mathbf{q}}$) on the right represent the sum of all the diagrams terminated by two electron lines $\beta\beta'$ (or phonon lines $\mathbf{q}'\mathbf{q}$) and are irreducible in the sense that any vertical cut on the horizontal part of the contour crosses more than two lines. The shaded figures W represent the sum of the diagrams which are irreducible in this sense and which have points only on the horizontal part of the contour. The quantities $F_{\beta\beta'}$ and $G_{\mathbf{q}'\mathbf{q}}$ do not include the factors $(s + i\omega_{\beta'\beta})^{-1}$ and $(s + i\omega_{\mathbf{q}'\mathbf{q}})^{-1}$, corresponding to the cutting of two right lines. The W's do not include the factors corresponding to the cutting of two pairs of lines joined to W from the left and from the right (pair 5), nor the factors corresponding to the pair on the left (pair 1).

The system shown graphically in Fig. 2 has the form

$$f_{\beta'\beta} = F_{\beta'\beta}(s + i\omega_{\beta'\beta})^{-1} + \sum_{\gamma'\gamma} W^{e}_{(\beta'\beta)(\gamma'\gamma)} f_{\gamma'\gamma}(s + i\omega_{\beta'\beta})^{-1} + \sum_{\mathbf{q}'\mathbf{q}} W^{ef}_{(\beta'\beta)(\mathbf{q}'\mathbf{q})} g_{\mathbf{q}'\mathbf{q}}(s + i\omega_{\beta'\beta})^{-1},$$

$$g_{\mathbf{q}'\mathbf{q}} = G_{\mathbf{q}'\mathbf{q}}(s + i\omega_{\mathbf{q}'\mathbf{q}})^{-1} + \sum_{\beta'\beta} W^{fe}_{(\mathbf{q}'\mathbf{q})(\beta'\beta)} f_{\beta'\beta}(s + i\omega_{\mathbf{q}'\mathbf{q}})^{-1} + \sum_{\mathbf{r}'\mathbf{r}} W^{f}_{(\mathbf{q}'\mathbf{q})(\mathbf{r}'\mathbf{r})} g_{\mathbf{r}'\mathbf{r}}(s + i\omega_{\mathbf{q}'\mathbf{q}})^{-1}. \tag{1.3}$$

The W's represent the kernels of the "collision integrals" and have the meaning of transition probabilities. Our problem reduces thus to the calculation of F, G, and W. We require an accuracy to V^2, so that the corresponding diagrams will have two vertices.

Figures 3 — 6 show typical diagrams for W. The diagrams W^{fe} are constructed similar to W^{ef}, and therefore only one is shown. In the diagonal components of W ($W^{e}_{(\beta\beta)(\gamma\gamma)}$, $W^{ef}_{(\beta\beta)(\mathbf{qq})}$ etc.) the principal values of the integrals involved cancel out and only the delta functions remain. As regards the nondiagonal elements of W, we shall see later on that in the approximation of interest to us (the lowest approximation that does not vanish in V) we can neglect the principal values in these elements.

The diagrams contained in F and G, which have vertices only on the vertical parts of the contour, do not contain irreversibilities (the parameter s) and correspond to renormalization of the spectrum of the system as a result of the interaction V. Further, in analogy with the statements made concerning W, we shall discard in F and G the principal values of the integrals and retain only the delta functions, since F and G represent

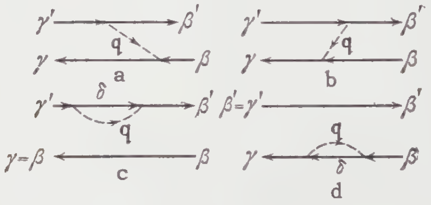

FIG. 3. Diagrams for W^e.

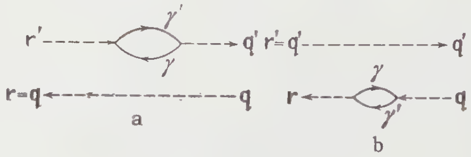

FIG. 4. Diagrams for W^f.

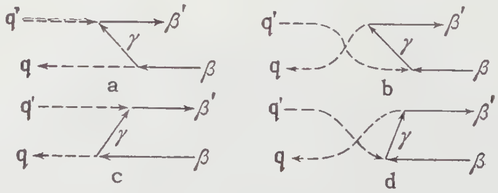

FIG. 5. Diagrams for W^{ef}.

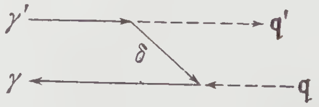

FIG. 6. Diagram for W^{fe}.

an inhomogeneity in the system of transport equations, and the discarded terms would yield only small additive corrections which are of no interest to us. Since diagrams in F and G with both vertices on the horizontal part of the contour are reducible, the only remaining diagrams are those indicated in Figs. 7 — 9, in which one point is on the vertical and one on the horizontal part of the contour. Figure 7 shows the free term F in the absence of interaction. The sum of the diagrams for F and G must be multiplied by $e(\mathbf{E}\cdot\mathbf{v})_{\alpha\alpha'}$ and summed over all α and α'.

We have included in the figures only the diagrams necessary to understand the method. To all the diagrams indicated for W, F, and G we must add those obtained by transferring the points from the upper horizontal section to the lower one (or vice versa). With this, it is necessary to take into account, in accordance with the rules indicated above, the change in the sign, the change in the number of intersections, and the change in the "regularity" of the incoming (or outgoing) lines to the right of the point. The "regularity" of the lines to the left does not change in such a transfer. As a result, the diagrams in which all the lines arrive at a point on the horizontal part from the left can be disregarded. Further, it is necessary to sum over the two possible directions of the phonon lines with dummy indices. It goes without saying that it is necessary to sum over the dummy indices in all diagrams.

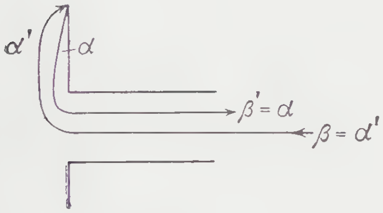

FIG. 7. Diagram for F without interaction.

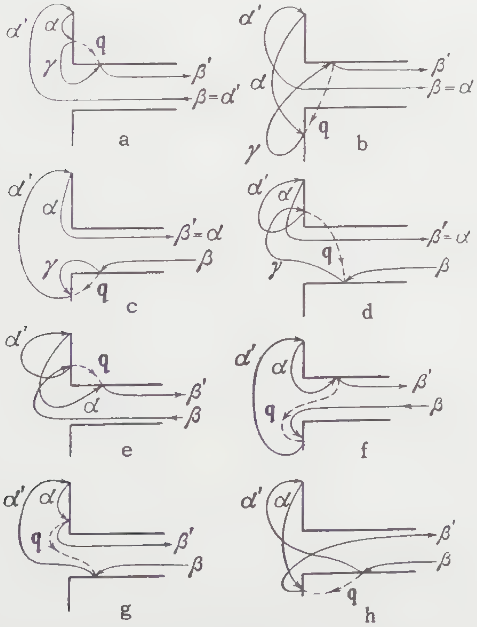

FIG. 8. Diagrams for F in the V^2 approximation.

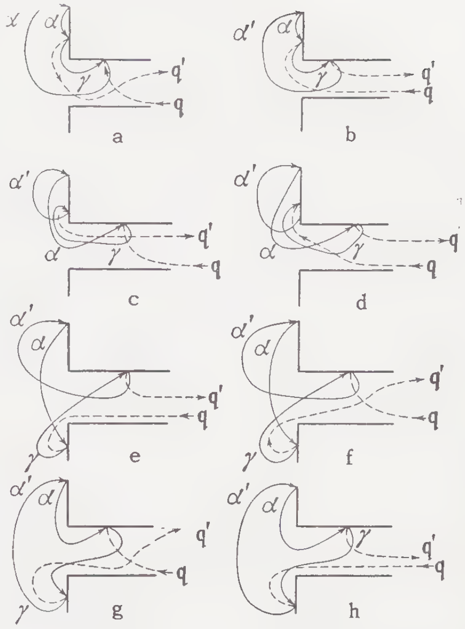

FIG. 9. Diagrams for G in the V^2 approximation.

Using the general form of the wave functions (1.1), and also the fact that $\mathbf{v}_{\alpha\alpha'}$ is diagonal over $\mathbf{P}$ $[\mathbf{P} = (P_y, P_z)]$, it is easy to conclude that F is diagonal over $\mathbf{P}$ and is independent of P_y while G is diagonal over $\mathbf{q}$. Further, it is easy to show that

$$W^{e}_{(\mathbf{PP})(\mathbf{P'P''})} \sim \delta_{\mathbf{P'P''}}, \qquad W^{ef}_{(\mathbf{P'P})(\mathbf{qq})} \sim \delta_{\mathbf{P'P}},$$
$$\int dP_y W^{fe}_{(\mathbf{q'q})(\mathbf{PP})} \sim \delta_{\mathbf{q'q}}, \qquad W^{f}_{(\mathbf{qq})(\mathbf{r'r})} \sim \delta_{\mathbf{r'r}}.$$

In addition, all the W are independent of P_y.

It follows from the properties of W that the matrix elements f, diagonal over $\mathbf{P}$, and the diagonal matrix elements g, are not "coupled" in the equations with matrix elements f and g which are not diagonal over $\mathbf{P}$ (or $\mathbf{q}$). But then it follows from the aforementioned properties of F and G that $f_{(n\mathbf{P})(n'\mathbf{P}')} \sim \delta_{\mathbf{PP}'}$ and $g_{\mathbf{q}'\mathbf{q}} \sim \delta_{\mathbf{q}'\mathbf{q}}$. Since neither F nor W depend on P_y, f is also independent of P_y, and we can introduce $f_{(n\mathbf{P})(n'\mathbf{P}')} = f_{nn'}(P_z)$. The foregoing properties are obviously the consequence of the spatial homogeneity of the problem, and take place only in the absence of spatial dispersion.

Calculating W with allowance for the approximations listed above, we obtain

$$
\begin{aligned}
W^{e}_{(\beta'\beta)(\alpha'\alpha)} = {} & \pi\hbar^{-2}\sum_{\mathbf{q}}|c_{\mathbf{q}}|^2 J^{*}_{\alpha'\beta'}J_{\alpha\beta}\{\delta(\omega_{\beta'\alpha}+\omega_{\mathbf{q}}) \\
& \times[(1-n_{\beta'})(N_{\mathbf{q}}+1)+n_{\beta'}N_{\mathbf{q}}]+\delta(\omega_{\beta'\alpha}-\omega_{\mathbf{q}})[(1-n_{\beta'})N_{\mathbf{q}} \\
& +n_{\beta'}(N_{\mathbf{q}}+1)]+\delta(\omega_{\beta\alpha'}+\omega_{\mathbf{q}})[(1-n_{\beta})(N_{\mathbf{q}}+1)+n_{\beta}N_{\mathbf{q}}] \\
& +\delta(\omega_{\beta\alpha'}-\omega_{\mathbf{q}})[(1-n_{\beta})N_{\mathbf{q}}+n_{\beta}(N_{\mathbf{q}}+1)]\} \\
& -\pi\hbar^{-2}\delta_{\beta\alpha}\sum_{\mathbf{q}}\sum_{\gamma}|c_{\mathbf{q}}|^2 J^{*}_{\alpha'\gamma}J_{\beta'\gamma}\{\delta(\omega_{\beta\gamma}-\omega_{\mathbf{q}}) \\
& \times[(1-n_{\gamma})(N_{\mathbf{q}}+1)+n_{\gamma}N_{\mathbf{q}}] \\
& +\delta(\omega_{\beta\gamma}+\omega_{\mathbf{q}})[(1-n_{\gamma})N_{\mathbf{q}}+n_{\gamma}(N_{\mathbf{q}}+1)]\} \\
& -\pi\hbar^{-2}\delta_{\beta'\alpha'}\sum_{\mathbf{q}}\sum_{\gamma}|c_{\mathbf{q}}|^2 J_{\alpha\gamma}J^{*}_{\beta\gamma}\{\delta(\omega_{\beta'\gamma}-\omega_{\mathbf{q}}) \\
& \times[(1-n_{\gamma})(N_{\mathbf{q}}+1)+n_{\gamma}N_{\mathbf{q}}] \\
& +\delta(\omega_{\beta'\gamma}+\omega_{\mathbf{q}})[(1-n_{\gamma})N_{\mathbf{q}}+n_{\gamma}(N_{\mathbf{q}}+1)]\}, \qquad (1.4a)
\end{aligned}
$$

$$
\begin{aligned}
W^{ef}_{(\beta'\beta)(\mathbf{qq})} = {} & \pi\hbar^{-2}|c_{\mathbf{q}}|^2\sum_{\gamma}\{(n_{\gamma}-n_{\beta})[\delta(\omega_{\gamma\beta}+\omega_{\mathbf{q}})J_{\beta'\gamma}J^{*}_{\beta\gamma} \\
& +\delta(\omega_{\gamma\beta}-\omega_{\mathbf{q}})J_{\gamma\beta}J^{*}_{\gamma\beta'}]+(n_{\gamma}-n_{\beta'})[\delta(\omega_{\gamma\beta'}+\omega_{\mathbf{q}})J_{\beta'\gamma}J^{*}_{\beta\gamma} \\
& +\delta(\omega_{\gamma\beta'}-\omega_{\mathbf{q}})J_{\gamma\beta}J^{*}_{\gamma\beta'}]\}, \qquad (1.4b)
\end{aligned}
$$

$$
\begin{aligned}
W^{fe}_{(\mathbf{qq})(\beta'\beta)} = {} & \pi\hbar^{-2}|c_{\mathbf{q}}|^2\sum_{\gamma}\{J_{\beta\gamma}J^{*}_{\beta'\gamma}[(1+N_{\mathbf{q}})(1-n_{\gamma}) \\
& +N_{\mathbf{q}}n_{\gamma}]\delta(\omega_{\beta'\gamma}-\omega_{\mathbf{q}}) \\
& -J_{\gamma\beta'}J^{*}_{\gamma\beta}[(1-n_{\gamma})N_{\mathbf{q}}+n_{\gamma}(N_{\mathbf{q}}+1)]\delta(\omega_{\beta'\gamma}+\omega_{\mathbf{q}}) \\
& +J_{\beta\gamma}J^{*}_{\beta'\gamma}[(1+N_{\mathbf{q}})(1-n_{\gamma})+N_{\mathbf{q}}n_{\gamma}]\delta(\omega_{\beta\gamma}-\omega_{\mathbf{q}}) \\
& -J_{\gamma\beta'}J^{*}_{\gamma\beta}[(1-n_{\gamma})N_{\mathbf{q}}+n_{\gamma}(N_{\mathbf{q}}+1)]\delta(\omega_{\beta\gamma}+\omega_{\mathbf{q}})\}. \qquad (1.4c)
\end{aligned}
$$

We do not write out W^f, since this operator is obtained by linearizing the usual integrals for collisions between phonons and equilibrium electrons, for collisions between phonons, or collisions with defects. As regards the operator of collisions between the electrons and the defects, this operator is obtained in the Born approximation from the one written out above by discarding the factors $N_{\mathbf{q}}$ or $N_{\mathbf{q}}+1$ and the addend $\omega_{\mathbf{q}}$ from the arguments of the delta function, and replacing $|c_{\mathbf{q}}|^2$ by $|V_{ed}(\mathbf{q})|^2 N_d$ (N_d is the number of defects).

In the longitudinal case, when $\mathbf{E} = E_z$ and $(\mathbf{E}\cdot\mathbf{v})_{\alpha\alpha'} \sim \delta_{\alpha\alpha'}$ so that $F_{\beta'\beta}$ which is diagonal in the zeroth approximation (sufficient for this case) of the interaction, we obtain, after taking the inverse Laplace transform in the variable σ, a system of transport equations of the ordinary form for diagonal matrices (occupation numbers) f and g.*

In the transverse case (for example $\mathbf{E} = E_x$), if the motion of the electron along the x axis is finite (i.e., we do not consider open trajectories[6]), then $(\mathbf{E}\cdot\mathbf{v})_{\alpha\alpha'} = 0$.

It is convenient to present F in the form

$$
F_{\beta'\beta} = F^{0}_{\beta'\beta} - \sum_{\gamma'\gamma} W^{e}_{(\beta'\beta)(\gamma'\gamma)}F^{0}_{\gamma'\gamma}(i\omega_{\gamma'\gamma})^{-1} + \Delta F_{\beta'\beta}; \qquad (1.5)
$$

$$
F^{0}_{\beta'\beta} = eEv^{(x)}_{\beta'\beta}(\hbar\omega_{\beta'\beta})^{-1}(n_{\beta}-n_{\beta'}), \qquad (1.6a)
$$

$$
\begin{aligned}
\Delta F_{\beta'\beta} = {} & -\pi eE(i\hbar^2 T)^{-1}\sum_{\gamma'\gamma\mathbf{q}}|c_{\mathbf{q}}|^2 N_{\mathbf{q}}\{v^{(x)}_{\gamma'\beta}\omega^{-1}_{\gamma'\beta} \\
& \times[J^{*}_{\gamma'\gamma}J_{\beta'\gamma}\delta(\omega_{\gamma\beta}+\omega_{\mathbf{q}})n_{\gamma}(1-n_{\beta}) \\
& +J^{*}_{\gamma\beta'}J_{\gamma\gamma'}\delta(\omega_{\gamma\beta}-\omega_{\mathbf{q}})n_{\beta}(1-n_{\gamma})] \\
& +v^{(x)}_{\beta'\gamma}\omega^{-1}_{\beta'\gamma}[J^{*}_{\beta\gamma'}J_{\gamma\gamma'}\delta(\omega_{\gamma'\beta'}+\omega_{\mathbf{q}})n_{\gamma'}(1-n_{\beta'}) \\
& +J^{*}_{\gamma'\gamma}J_{\gamma'\beta}\delta(\omega_{\gamma'\beta'}-\omega_{\mathbf{q}})n_{\beta'}(1-n_{\gamma'})] \\
& -v^{(x)}_{\gamma'\gamma}\omega^{-1}_{\gamma'\gamma}[J^{*}_{\beta\gamma}J_{\beta'\gamma'}\delta(\omega_{\gamma'\beta}+\omega_{\mathbf{q}})n_{\gamma'}(1-n_{\beta}) \\
& +J^{*}_{\gamma'\beta'}J_{\gamma\beta}\delta(\omega_{\gamma'\beta}-\omega_{\mathbf{q}})n_{\beta}(1-n_{\gamma'}) \\
& +J^{*}_{\beta\gamma}J_{\beta'\gamma'}\delta(\omega_{\gamma\beta'}+\omega_{\mathbf{q}})n_{\gamma}(1-n_{\beta'}) \\
& +J^{*}_{\gamma'\beta'}J_{\gamma\beta}\delta(\omega_{\gamma\beta'}-\omega_{\mathbf{q}})n_{\beta'}(1-n_{\gamma})]\}. \qquad (1.6b)
\end{aligned}
$$

Noting that

$$
(i\omega_{\gamma'\beta})^{-1}v^{(x)}_{\gamma'\beta} = x_{\gamma'\beta} \qquad (\gamma' \neq \beta),
$$

$$
\begin{aligned}
\sum_{\gamma'\neq\beta} x_{\gamma'\beta}J^{*}_{\gamma'\gamma}(\mathbf{q}) & = \sum_{\gamma'\neq\beta} x_{\gamma'\beta}J_{\gamma\gamma'}(-\mathbf{q}) \\
& = \sum_{\gamma'} J_{\gamma\gamma'}(-\mathbf{q})x_{\gamma'\beta} - x_{\beta\beta}J_{\gamma\beta}(-\mathbf{q}) \\
& = (e^{-i\mathbf{qr}/\hbar}x)_{\gamma\beta} - x_{\beta\beta}J^{*}_{\beta\gamma}(\mathbf{q}),
\end{aligned}
$$

and carrying out analogous transformations in the remaining terms of (1.6), we obtain, recognizing that the operators $\exp(i\mathbf{q}\cdot\mathbf{r})$ and x commute, and also that $J_{\gamma\gamma'}(-\mathbf{q}) = J^{*}_{\gamma\gamma'}(\mathbf{q})$ and $\omega(\mathbf{q}) = \omega(-\mathbf{q})$,

$$
\begin{aligned}
\Delta F_{\beta'\beta} = {} & \pi eE(\hbar^2 T)^{-1}\sum_{\gamma\mathbf{q}}|c_{\mathbf{q}}|^2 N_{\mathbf{q}}J^{*}_{\beta\gamma}J_{\beta'\gamma} \\
& \times\{X_{\beta\gamma}[\delta(\omega_{\gamma\beta}+\omega_{\mathbf{q}})n_{\gamma}(1-n_{\beta}) \\
& +\delta(\omega_{\gamma\beta}-\omega_{\mathbf{q}})n_{\beta}(1-n_{\gamma})] \\
& +X_{\beta'\gamma}[\delta(\omega_{\gamma\beta'}+\omega_{\mathbf{q}})n_{\gamma}(1-n_{\beta'}) \\
& +\delta(\omega_{\gamma\beta'}-\omega_{\mathbf{q}})n_{\beta'}(1-n_{\gamma})]\}. \qquad (1.7)
\end{aligned}
$$

Here $X_{\alpha\beta} = x_{\alpha\alpha} - x_{\beta\beta}$.

Analogously we obtain

$$
\begin{aligned}
G_{\mathbf{q}} = {} & -\sum_{\beta\beta'} W^{fe}_{(\mathbf{qq})(\beta'\beta)}F^{0}_{\beta'\beta}(i\omega_{\beta'\beta})^{-1} \\
& +|c_{\mathbf{q}}|^2 2\pi eE(\hbar^2 T)^{-1}\sum_{\beta\beta'} N_{q}n_{\beta}(1-n_{\beta'}) \\
& \times\delta(\omega_{\beta'\beta}-\omega_{\mathbf{q}})|J_{\beta'\beta}|^2 X_{\beta'\beta}. \qquad (1.8)
\end{aligned}
$$

We note that the matrices f and g are Hermitian, because

$$
\begin{gathered}
W^{e}_{(\beta'\beta)(\gamma'\gamma)} = W^{e*}_{(\beta\beta')(\gamma\gamma')}, \quad W^{ef}_{(\beta'\beta)(\mathbf{qq})} = W^{ef*}_{(\beta\beta')(\mathbf{qq})}, \\
W^{fe}_{(\mathbf{qq})(\beta'\beta)} = W^{fe*}_{(\mathbf{qq})(\beta\beta')}, \quad W^{f}_{(\mathbf{qq})(\mathbf{rr})} = W^{f*}_{(\mathbf{qq})(\mathbf{rr})}, \\
F_{\beta'\beta} = F^{*}_{\beta\beta'}, \quad G_{\mathbf{q}} = G^{*}_{\mathbf{q}}.
\end{gathered}
$$

---

*Subject to the condition $\omega\tau \gg 1$ (in the notation of Sec. 2).

2. CASE OF STRONG ($\omega\tau \gg 1$) TRANSVERSE MAGNETIC FIELD. DERIVATION OF A SYSTEM OF EQUATIONS FOR THE DIAGONAL PARTS OF THE MATRICES f AND g

In the case of a strong magnetic field transverse to the electric field, we can expand the non-diagonal part of the matrix f in powers of $(\omega\tau)^{-1} \ll 1$, where ω is the Larmor frequency of the electrons and τ is their relaxation time. This corresponds to assuming W and ΔF in (1.5) to be small. In this case it is sufficient to calculate the values of $W \sim \tau^{-1}$ in the lower approximation in V (in the case of scattering by impurities — in the lower approximation in the concentration).

Diagrams of higher order contain parameters of the type $\hbar(\tau T)^{-1}$ or $\hbar(\tau\zeta)^{-1}$ (ζ is the chemical potential of the electrons), which are small in most cases, and can be disregarded here. As regards F (or G), the addition of points on the horizontal parts of the corresponding diagrams obviously yields corrections of the same type as due to the addition of points in the diagrams of W. The addition of points on the vertical parts corresponds, on the other hand, to thermodynamic perturbation theory (i.e., to the expansion of the operator $\hat{\rho}_0 = Z^{-1}\exp[-\beta(\hat{\mathcal{H}}_0 + \hat{V} - \zeta N)]$ in powers of $\hat{V}$), and therefore cannot lead to either appearance or disappearance of currents, but can only give rise to corrections to these quantities, which are of no interest to us. Therefore, at least for the calculation of F (or G), it is sufficient to consider only diagrams with a single point on the vertical part, and in our case also with a single point on the horizontal portion, as we have done in Sec. 1.

We shall now show that before we expand in powers of W and ΔF [i.e., in $(\omega\tau)^{-1}$], we must solve the system of integral equations for the matrix g and the diagonal part of the matrix f. We introduce the symbol f^d and f^n for the diagonal and non-diagonal parts of f, and analogously for other matrices. Then the system (1.3) can be represented in the form

$$i\omega_{\beta'\beta}f^{(n)}_{\beta'\beta} = F^{0\,(n)}_{\beta'\beta} - \Big(\sum_{\gamma'\gamma} W^{e}_{(\beta'\beta)\,(\gamma'\gamma)} F^{0\,(n)}_{\gamma'\gamma}(i\omega_{\gamma'\gamma})^{-1}\Big)^{(n)} + \Delta F^{(n)}_{\beta'\beta}$$
$$+ \Big(\sum_{\gamma'\gamma} W^{e}_{(\beta'\beta)(\gamma'\gamma)} f^{(n)}_{\gamma'\gamma}\Big)^{(n)} + \Big(\sum_{\gamma} W^{e}_{(\beta'\beta)\,(\gamma\gamma)}\, f^{d}_{\gamma\gamma}\Big)^{(n)}$$
$$+ \Big(\sum_{q} W^{ef}_{(\beta'\beta)\,(\mathbf{qq})}\ g_{\mathbf{q}}\Big)^{(n)}, \tag{2.1a}$$

$$-\sum_{\gamma'\gamma} W^{e}_{(\beta\beta)\,(\gamma'\gamma)}\, F^{0\,(n)}_{\gamma'\gamma}\,(i\omega_{\gamma'\gamma})^{-1} + \Delta F_{\beta\beta} + \sum_{\gamma'\gamma} W^{e}_{(\beta\beta)\,(\gamma'\gamma)}\, f^{(n)}_{\gamma'\gamma}$$
$$+ \sum_{\gamma} W^{e}_{(\beta\beta)\,(\gamma\gamma)}\, f^{d}_{\gamma\gamma} + \sum_{\mathbf{q}} W^{ef}_{(\beta\beta)\,(\mathbf{qq})}\, g_{\mathbf{q}} = 0, \tag{2.1b}$$

$$-\sum_{\gamma'\gamma} W^{fe}_{(\mathbf{qq})\,(\gamma'\gamma)}\, F^{0\,(n)}_{\gamma'\gamma}\,(i\omega_{\gamma'\gamma})^{-1} + \Delta G_{\mathbf{q}} + \sum_{\beta'\beta} W^{fe}_{(\mathbf{qq})\,(\beta'\beta)}\, f^{(n)}_{\beta'\beta}$$
$$+ \sum_{\beta} W^{fe}_{(\mathbf{qq})\,(\beta\beta)}\, f^{d}_{\beta\beta} + \sum_{\mathbf{r}} W^{f}_{(\mathbf{qq})\,(\mathbf{rr})}\, g_{\mathbf{r}} = 0. \tag{2.1c}$$

In the zeroth approximation $f^{0(n)}_{\beta'\beta} = F^{0(n)}_{\beta'\beta}(i\omega\beta'\beta)^{-1}$, and f^{0d}, and g^0 remain indeterminate. To determine these we must substitute in (2.1b) and (2.1c) the values of $f^{0(n)}$. We then obtain a system of integral equations for f^{0d} and g^0, which is more convenient to write down by introducing the quantities φ_α and $\gamma_{\mathbf{q}}$, related to f^d and g by the equations

$$f^{d}_{\alpha\alpha} = -(\partial n_\alpha/\partial\varepsilon_\alpha)\,\varphi_\alpha, \qquad g_{\mathbf{q}} = -[\partial N_q/\partial(\hbar\omega_q)]\,\gamma_{\mathbf{q}}. \tag{2.2}$$

φ^0_α and $\gamma^0_{\mathbf{q}}$ satisfy the system of equations

$$2\pi(\hbar^2T)^{-1}\sum_{\alpha\mathbf{q}} |c_{\mathbf{q}}|^2 |J_{\beta\alpha}|^2 [\delta(\omega_{\alpha\beta}+\omega_{\mathbf{q}})\, n_\alpha(1-n_\beta)$$
$$+\delta(\omega_{\alpha\beta}-\omega_{\mathbf{q}})\, n_\beta(1-n_\alpha)]$$
$$\times[eEX_{\beta\alpha}-(\varphi^0_\beta-\varphi^0_\alpha)+\gamma^0_{\mathbf{q}}]\, N_{\mathbf{q}}+2\pi(\hbar^2T)^{-1}$$
$$\times\sum_{\alpha\mathbf{q}}|V_{ed}(\mathbf{q})|^2\,|J_{\beta\alpha}|^2\,\delta(\omega_{\alpha\beta})\, n_\beta(1-n_\beta)$$
$$\times[eEX_{\beta\alpha}-(\varphi^0_\beta-\varphi^0_\alpha)]\, N_d = 0;$$
$$2\pi(\hbar^2T)^{-1}\sum_{\alpha\beta}|c_{\mathbf{q}}|^2\, n_\alpha(1-n_\beta)\, N_{\mathbf{q}}\delta(\omega_{\beta\alpha}-\omega_{\mathbf{q}})\,|J_{\beta\alpha}|^2$$
$$\times[eEX_{\alpha\beta}-(\varphi^0_\alpha-\varphi^0_\beta)-\gamma^0_{\mathbf{q}}]+\hat{W}_{ff}(\gamma_{\mathbf{q}})+\hat{W}_{fd}(\gamma_{\mathbf{q}}) = 0. \tag{2.3}$$

Here W_{ff} and W_{fd} are the operators of the phonon-phonon and phonon-defect collisions, respectively.

Substituting the solution of these equations into (2.1a) in the first approximation, we obtain for $f^{1(n)}$

$$f^{1\,(n)}_{\beta'\beta} = \pi(i\omega_{\beta'\beta}\,\hbar^2T)^{-1}\sum_{\alpha q} J^*_{\beta\alpha}J_{\beta'\alpha}\{|c_{\mathbf{q}}|^2 N_{\mathbf{q}}\{[\delta(\omega_{\alpha\beta}+\omega_{\mathbf{q}})n_\alpha$$
$$\times(1-n_\beta)+\delta(\omega_{\alpha\beta}-\omega_{\mathbf{q}})\, n_\beta(1-n_\alpha)]\,[eEX_{\beta\alpha}-(\varphi^0_\beta-\varphi^0_\alpha)$$
$$+\gamma^0_{\mathbf{q}}]+[\delta(\omega_{\alpha\beta'}+\omega_{\mathbf{q}})\, n_\alpha(1-n_{\beta'})$$
$$+\delta(\omega_{\alpha\beta'}-\omega_{\mathbf{q}})\, n_{\beta'}(1-n_\alpha)]\,[eEX_{\beta'\alpha}-(\varphi^0_{\beta'}-\varphi^0_\alpha)$$
$$+\gamma^0_{\mathbf{q}}]\}+|V_{ed}(\mathbf{q})|^2 N_d\ \{\delta(\omega_{\alpha\beta})\, n_\beta(1-n_\beta)$$
$$\times\ [eEX_{\beta\alpha}-(\varphi^0_\beta-\varphi^0_\alpha)]+\delta(\omega_{\alpha\beta'})\, n_{\beta'}(1-n_{\beta'})$$
$$\times\ [eEX_{\beta'\alpha}-(\varphi^0_{\beta'}-\varphi^0_\alpha)]\}\}. \tag{2.4}$$

The resultant values $f^{1(n)}$ are again inserted in (2.1b) and (2.1c), after which we obtain f^d and g in the next approximation. This process of successive approximation corresponds to expansion in powers of the parameter $(\omega\tau)^{-1}$.

It is easy to see that the equations for f^d and g in the different approximations differ only in the form of the inhomogeneity. We note that the nondiagonal elements of W (i.e., $W^{e}_{(\beta'\beta)(\gamma'\gamma)}$, in which the equations $\beta' = \beta$ and $\gamma' = \gamma$ are not satisfied simultaneously, and analogously for W^{ef} and W^{fe}) enter into the integral equations only as inhomogeneities, and we can therefore discard in these quantities the principal values of the integrals, as in the case of F (or G). The same pertains also to the terms with W in (2.4).

The integral equations (2.3) have the form of ordinary transport equations, and the singularities of the problem are due only to the inhomogeneity. Since the wave functions (1.1) depend on the argument $(c/eH)p_y$, where $p_y = P_y - (eH/c)x$, the matrix elements are $x_{\alpha\alpha} = (c/eH)[(P_y)_{\alpha\alpha} - (p_y)_{\alpha\alpha}]$. The quantity $(p_y)_{\alpha\alpha}$ obviously has the meaning of the average value of the momentum p_y on the quasi-classical orbit $\epsilon(\mathbf{p}) = \text{const}$, $p_z = \text{const}$.

If the crystal has mirror symmetry with respect to the xz plane, then the first equation in (2.3) is satisfied by $\varphi_\alpha = u_e(p_y)_{\alpha\alpha}$, where $u_e = -cE/H$ is the electron drift (Hall) velocity, for in this case $(P_y)_{\beta\beta} - (P_y)_{\alpha\alpha} = q_y$, $\gamma(q_y) = -\gamma(-q_y)$. Then in the second equation in (2.3) the term corresponding to the phonon-electron interaction has the form $[u_e q_y - \gamma(\mathbf{q})]\tau_{fe}^{-1}$, corresponding precisely to the phenomenological theory if $\gamma(\mathbf{q}) = u_f q_y$, where u_f has the meaning of the phonon drift velocity. (Here τ_{fe} is the ordinary relaxation time of phonons on equilibrium electrons).

In the case of an isotropic spectrum $(p_y)_{\alpha\alpha} = 0$ and therefore $\varphi_\alpha = 0$. In the particular case of an isotropic quadratic spectrum, neglecting the entrainment of electrons by the phonons, we obtain the results of Adams and Holstein[7] for $f^{1n}_{\alpha\alpha'}$, and our matrix $n_\alpha\delta_{\alpha\alpha'} + f^{0n}_{\alpha\alpha'}$ is identical with the matrix $\rho(0)$ of this paper, written in the $\hat{\mathcal{H}}_0$ representation.

In the absence of defects (and also in the absence of Umklapp processes) the system (2.3) is satisfied by the values $\varphi_\alpha = u_e(p_y)_{\alpha\alpha}$ and $\gamma(\mathbf{q}) = u_e q_y$, corresponding to an over-all drift of the electron-phonon system with a velocity u_e [the term $W^{ff}(\gamma_q)$ also vanishes in this case]. We now get in (2.4) $f^{1(n)} = 0$, which is understandable, for in the absence of scattering by the defects the total momentum of the system is conserved, corresponding formally to $\tau = \infty$.

We also note the following. Let the phonons be in equilibrium. Then the equation for φ_α has the form $\hat{L}\chi = 0$, where $\chi_\alpha = eEx_{\alpha\alpha} - \varphi_\alpha$. Obviously the solution $\chi_\alpha \equiv 0$ (from which follows $f^{1(n)}_{\beta'\beta} \equiv 0$) corresponds merely to a Boltzmann charge distribution in a potential electric field, not accompanied by an irreversible current. The nontrivial solution for χ is determined, in view of the homogeneity of the equation, accurate to a constant factor, which can be defined in such a manner as to make $\varphi_\alpha = eEx_{\alpha\alpha} - \chi_\alpha$ independent of P_y (which is possible, as was already noted in Sec. 1). With this, since $f_{\alpha\alpha'}$ is diagonal over P_z and is independent of P_y, and the wave functions have the form (1.1), the charge density

$$e\sum_{\alpha\alpha'} f_{\alpha\alpha'}\psi^*_{\alpha'}(x)\,\psi_\alpha(x)$$

$$= \sum_{nn'P_z} f_{nn'}(P_z) \sum_{P_y} \psi^*_{n'P_z}(x - x_0)\,\psi_{nP_z}(x - x_0)$$

is independent of x.

Knowing the functions φ and γ in the zeroth approximation, we can determine $f^{1(n)}$ and consequently, the component of the current density along the electric field

$$j^{(x)} = \frac{2e}{V}\sum_{\beta'\beta} v^{(x)}_{\beta\beta'} f^{1(n)}_{\beta'\beta}$$

(V is the volume of the sample and the factor 2 corresponds to the two spin orientations). After transformations analogous to those used to transform (1.6) into (1.7), we obtain

$$\begin{aligned} j^{(x)} = \frac{1}{V}\frac{4\pi e}{\hbar^2 T}\sum_{\alpha\beta\mathbf{q}} |J_{\beta\alpha}|^2 \{ |c_{\mathbf{q}}|^2 N_{\mathbf{q}} n_\alpha (1 - n_\beta)\,\delta(\omega_{\alpha\beta} + \omega_{\mathbf{q}}) \\ \times X_{\beta\alpha}[eEX_{\beta\alpha} - (\varphi^0_\beta - \varphi^0_\alpha) + \gamma^0(\mathbf{q})] \\ + |V_{ed}(\mathbf{q})|^2 N_d n_\beta (1 - n_\beta)\,\delta(\omega_{\alpha\beta}) \\ \times X_{\beta\alpha}[eEX_{\beta\alpha} - (\varphi^0_\beta - \varphi^0_\alpha)]\}. \end{aligned} \tag{2.5}$$

The formula obtained by Adams and Holstein[7] is a particular case of this formula.

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EMISSION OF LOW ENERGY γ QUANTA BY ELECTRONS SCATTERED ON PROTONS

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The emission of low energy γ quanta in ep scattering is considered. It is shown that the first terms of the expansion of the amplitude in powers of the photon energy can be expressed through the electromagnetic form factors of the proton. The differential cross section for the process is derived in this approximation.

1. INTRODUCTION

THE study of elastic scattering of high-energy electrons by protons and nuclei is, at present, a basic source of information on the electromagnetic structure of nucleons. The experiments performed have made it possible to draw important conclusions on the behavior of the electromagnetic form factors of nucleons $F_1^{p;n}$ and $F_2^{p;n}$ as functions of the square of the four-momentum transfer. Great interest attaches also to other processes, the study of which will make it possible to obtain additional information on the form factors F_1 and F_2.[1,2] In the present article, we consider the emission of low energy bremsstrahlung in ep scattering. It will be shown that the first two terms of the expansion of the differential cross section for this process in powers of the photon energy are expressed through the electromagnetic form factors of the proton and their derivatives with respect to the momentum transfer.

To obtain the amplitude of the process, we employ the general method of considering the radiation of low-energy γ quanta suggested by Low.[3,4] Following this method, we first consider that part of the amplitude of the process which has a pole at $k = 0$ (k is the photon momentum). It is then shown that the product of the exact renormalized vertex operator corresponding to the radiation of a real γ quantum and the exact renormalized propagation function of the nucleon is expressed, according to Ward's identity, to an accuracy of terms of the order $1/k$ and constants, through the charge, mass, and anomalous magnetic moment of a physical nucleon. On the basis of gauge invariance, we then find the part of the amplitude not containing the pole. In the concluding part of the article, we derive an expression for the differential cross section of the process.

2. AMPLITUDE OF THE PROCESS

The process of the radiation of γ quanta in the scattering of electrons by protons in the lowest-order approximation in e, but with allowance for strong interactions of the nucleon, is described by the diagrams of Figs. 1 and 2. In the figures, q and q′ denote the electron 4-momenta before and after the collision; p and p′ denote the proton momenta, and k is the γ-quantum momentum. It is obvious that the nucleon vertex part of Figs. 1a and 1b is determined by the form factors describing the elastic scattering of electrons by protons with a corresponding momentum transfer.

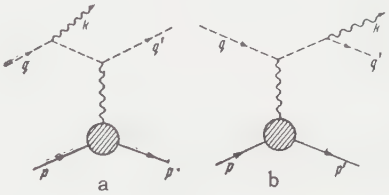

Fig. 1

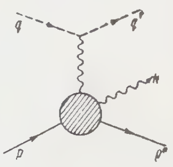

Fig. 2

We write the S matrix in the form

$$S(p', q', k; p, q) = -(2\pi)^4 i \left(M^2 m^2 / 2k_0 p_0' p_0 q_0' q_0\right)^{1/2} \varepsilon_\mu \times (T_\mu^{\mathrm{I}} + T_\mu^{\mathrm{II}})\, \delta(p' + q' + k - p - q), \quad (1)$$

where M and m are the masses of the proton and electron, ϵ_μ is the polarization vector of the photon, $p_0 = -ip_4$, $q_0 = -iq_4$, etc, and T_μ^{I} and T_μ^{II} represent the respective contributions of the diagrams of Figs. 1a, b and 2.

For T_μ^{I} we have

$$T_\mu^{\mathrm{I}} = \bar{u}(q')\left[ie\gamma_\mu \frac{1}{i\gamma(q'+k)+m} ie\gamma_\nu + ie\gamma_\nu \frac{1}{i\gamma(q-k)+m} ie\gamma_\mu\right] u(q)\ \bar{v}(p')\, ie\,[F_1((p'-p)^2)\gamma_\nu - \tfrac{1}{2}\mu_p M^{-1} F_2((p'-p)^2)\,\sigma_{\nu\rho}(p'-p)_\rho]\, v(p)\,(p'-p)^{-2}. \quad (2)$$

Here μ_p, F_1, and F_2 are the anomalous moment (in nuclear magnetons) and the electromagnetic form factors of the proton; $\sigma_{\mu\rho} = (1/2i)(\gamma_\mu\gamma_\rho - \gamma_\rho\gamma_\mu)$. The spinors u and v are normalized by the conditions $\bar{u}u = 1$, $\bar{v}v = 1$.

We shall now consider diagrams of the type in Fig. 2. We separate these diagrams into two classes A and B.[5] In class A we include all diagrams in which the vertex with the emission of a real quantum is connected with the remaining part of the diagram by a nucleon line. In class B we include all other diagrams. We write T^{II}_μ in the following way:

$$T^{II}_\mu = -\bar{u}(q')\, ie\gamma_\nu u(q)\, [T^A_{\nu\mu} + T^B_{\nu\mu}]\,(q'-q)^{-2}. \quad (3)$$

$T^A_{\nu\mu}$ and $T^B_{\nu\mu}$ describe the contribution of diagrams of classes A and B. For $T^A_{\nu\mu}$ we have the following expression:

$$T^A_{\nu\mu} = \bar{v}(p')\,[ie\Gamma_\mu(p',\ p'+k)\, S(p'+k)\, ie\Gamma_\nu(p'+k,\ p) + ie\Gamma_\nu(p',\ p-k)\, s(p-k)\, ie\Gamma_\mu(p-k,\ p)]\, v(p). \quad (4)$$

Here S and Γ are the exact renormalized propagation function and electromagnetic vertex operator of the proton.

We are interested in the radiation of low-energy γ quanta. We therefore expand the operators occurring in (4) in powers of k and limit ourselves to the first two terms of the expansion. We consider first $S(p-k)\Gamma_\mu(p-k,\ p)v(p)$. The general expression for the propagation function can be written in the form

$$S(t) = [i\gamma t G(t^2) + MF(t^2)]^{-1}. \quad (5)$$

The values of the functions $G(t^2)$ and $F(t^2)$ and their derivatives $F'(t^2)$ and $G'(t^2)$ at $t^2 = -M^2$ are, as is known,[5] related in the following way:

$$F = G, \qquad F + 2M^2(F' - G') = 1. \quad (6)$$

Expanding $S(p-k)$ in a series in powers of k, using (6), and retaining the first two terms of the expansion, we obtain

$$S(p-k) = -(2pkF)^{-1}\{(-i\gamma p + M)F + i\gamma kF - 2pk(-i\gamma pG' + MF') + 2pkM^2F(-i\gamma p + M)\,[2G'/M^2 + F'' - G'' + (F'^2 - G'^2)/F]\}. \quad (7)$$

The operator $\Gamma_\mu(t',\ t)$ has the transformation properties of a 4-vector and, under charge conjugation, transforms like a current vector:

$$C^{-1}\Gamma_\mu(t',\ t)\,C = -\Gamma^T_\mu(-t,\ -t'). \quad (8)$$

We thus obtain the expansion of Γ_μ in a series in powers of k:[4,5]

$$\Gamma_\mu(p-k,\ p) = \Gamma_\mu(p,\ p) - \tfrac{1}{2}k_\rho\partial\Gamma_\mu(p,\ p)/\partial p_\rho + pkM^{-3}G_1(p^2)\sigma_{\mu\rho}p_\rho + \tfrac{1}{2}M^{-1}G_2(p^2)\,\sigma_{\mu\rho}k_\rho + G_3(p^2)\,M^{-3}\sigma_{\lambda\rho}k_\lambda p_\rho p_\mu + \tfrac{1}{2}G_4(p^2)\,M^{-2}\varepsilon_{\mu\lambda\rho\delta}p_\lambda k_\rho\gamma_\delta\gamma_5. \quad (9)$$

The first two terms of this expansion are expressed through the functions F and G and their derivatives by means of Ward's identity:

$$\Gamma_\mu(p,\ p) = -i\partial S^{-1}(p)/\partial p_\mu. \quad (10)$$

The terms in (9) containing G_1 and G_3 do not make any contribution to $S(p-k)\Gamma_\mu(p-k,\ p)v(p)$ in the required order, since $(-i\gamma p + M)\sigma_{\mu\rho}p_\rho v(p) = 0$. Using also the relation

$$(-i\gamma p + M)\,\varepsilon_{\mu\lambda\rho\delta}p_\lambda k_\rho\gamma_\delta\gamma_5 v(p) = (-i\gamma p + M)(-M)\,\sigma_{\mu\rho}k_\rho v(p) \quad (11)$$

and discarding, by the Lorentz condition, terms proportional to k_μ, we obtain, to an accuracy of terms of the order 1/k and constants, the following expression:

$$S(p-k)\,\Gamma_\mu(p-k,\ p)\,v(p) = -(2pk)^{-1}[-2ip_\mu + (-i\gamma p + M)\,\sigma_{\mu\rho}k_\rho \times (F - 1 + G_2 - G_4)/2M + i(\gamma k)\gamma_\mu]\,v(p). \quad (12)$$

If we consider the scattering of a proton by an external constant magnetic field, then it can be shown, by means of (9), that the quantity $F - 1 + G_2 - G_4$ is equal to the anomalous magnetic moment of the proton μ_p.[5] It is obvious that expression (12) can also be written in the following form:

$$S(p-k)\,ie\Gamma_\mu(p-k,\ p)\,v(p) = \frac{1}{i\gamma(p-k) + M}\left(ie\gamma_\mu + \mu_p\frac{ie}{2M}\sigma_{\mu\rho}k_\rho\right)v(p). \quad (13)$$

Hence $S(p-k)ie\Gamma_\mu(p-k,\ p)v(p)$ is expressed through the charge, magnetic moment, and mass of a physical proton if we consider only terms of the order 1/k and constants.

We note that the exact expression (13) obtained by us is of the same form as the corresponding expression obtained in the lowest-order approximation of perturbation theory for a point proton with a Pauli anomalous magnetic moment. Similarly, it can be shown that, in the approximation we are considering,

$$\bar{v}(p')\,ie\Gamma_\mu(p',\ p'+k)\,S(p'+k) = \bar{v}(p')\left(ie\gamma_\mu + \mu_p\frac{ie}{2M}\sigma_{\mu\rho}k_\rho\right)\frac{1}{i\gamma(p'+k) + M}. \quad (14)$$

For what follows, it will be convenient to introduce the invariants

$$M_1^2 = -(p-k)^2 = M^2 + 2pk,$$
$$M_2^2 = -(p'+k)^2 = M^2 - 2p'k. \tag{15}$$

With an accuracy to terms of the order 1/k and constants, we have the relations

$$(-i\gamma(p-k) + M)/2pk$$
$$= (-i\gamma(p-k) + M_1)/2pk - 1/2M,$$
$$(-i\gamma(p'+k) + M)/2p'k$$
$$= (-i\gamma(p'+k) + M_2)/2p'k + 1/2M. \tag{16}$$

Noting that

$$\gamma(p-k)\,[-i\gamma(p-k)$$
$$+ M_1] = iM_1[-i\gamma(p-k) + M_1],$$
$$[-i\gamma(p'+k) + M_2]\,\gamma(p'+k)$$
$$= iM_2\,[-i\gamma(p'+k) + M_2], \tag{17}$$

we obtain the following expression for $T^A_{\nu\mu}$:

$$T^A_{\nu\mu} = \bar{v}(p')\Big[ie\Gamma^0_\nu(p', p-k)\frac{1}{i\gamma(p-k)+M}\Big(ie\gamma_\mu + \frac{ie}{2M}\mu_p\sigma_{\mu\rho}k_\rho\Big)$$
$$+ \Big(ie\gamma_\mu + \frac{ie}{2M}\mu_p\sigma_{\mu\rho}k_\rho\Big)\frac{1}{i\gamma(p'+k)+M}\, ie\Gamma_\nu(p'+k, p)\Big]v(p)$$
$$+ \bar{v}(p')\,\{[ie\Gamma_\nu(p', p-k)$$
$$- ie\Gamma^0_\nu(p', p-k)]\, ie\gamma_\mu/2M\}\, v(p)$$
$$+ \bar{v}(p')\,\{(ie\gamma_\mu/2M)\;[ie\Gamma_\nu(p'+k, p)$$
$$- ie\Gamma^0_\nu(p'+k, p)]\}\, v(p), \tag{18}$$

where

$$\Gamma^0_\nu(t', t) = a\gamma_\nu + b\sigma_{\nu\rho}(t'-t)_\rho + c\sigma_{\nu\rho}(t'+t)_\rho. \tag{19}$$

This operator is obtained from the general expression for the vertex part $\Gamma_\nu(t', t)$ by putting $i\sqrt{-t^2}$ in place of the operator γt on the right and $i\sqrt{-t'^2}$ in place of the operator $\gamma t'$ on the left. In formula (19), the quantities a, b, and c are functions of $-t^2$, $-t'^2$, and $(t'-t)^2$, where a and b do not change when t is replaced by t′, while c changes sign. We note that, for $t^2 = t'^2 = -M^2$, the functions a and b are equal to $F_1[(t'-t)^2]$ and $-(\mu_p/2M)\,F_2[(t'-t)^2]$, and the function c vanishes.

We now consider the contribution to the amplitude from diagrams of class B. The contribution of $T^B_{\nu\mu}$ of these diagrams, as $k \to 0$, tends to a constant, while the limit does not depend on the way in which k tends to zero.[3] The last two terms in (18) also have this property. We denote the sum of these terms and $T^B_{\nu\mu}$ by $M^{(2)}_{\nu\mu}$; by $M^{(1)}_{\nu\mu}$ we understand the sum of the first two terms:

$$M^{(1)}_{\nu\mu} = \bar{v}(p')\Big[ie\Gamma^0_\nu(p', p-k)$$
$$\times \frac{1}{i\gamma(p-k)+M}\Big(ie\gamma_\mu + \frac{ie}{2M}\mu_\mu\sigma_{\rho\rho}k_\rho\Big)$$
$$+ \Big(ie\gamma_\mu + \frac{ie}{2M}\mu_p\sigma_{\mu\rho}k_\rho\Big)$$
$$\times \frac{1}{i\gamma(p'+k)+M}\, ie\Gamma^0_\nu(p'+k, p)\Big]v(p). \tag{20}$$

Owing to guage invariance,

$$k_\mu(T^{\mathrm{I}}_\mu + T^{\mathrm{II}}_\mu) = 0. \tag{21}$$

It is readily seen from formula (2) that $k_\mu T^{\mathrm{I}}_\mu = 0$. Hence it follows from (20) and (21) that

$$k_\mu M^{(2)}_{\nu\mu} = -k_\mu M^{(1)}_{\nu\mu} = e\bar{v}(p')\,[ie\Gamma^0_\nu(p', p-k)$$
$$- ie\Gamma^0_\nu(p'+k, p)]\, v(p). \tag{22}$$

By the law of conservation of energy-momentum $(p' - p + k = q - q')$, the operators Γ^0_ν occurring in (20) and (22) can be written in the form $(\kappa = q - q')$

$$\Gamma^0_\nu(p', p-k) = a(M^2, M^2 + 2pk, \varkappa^2)\,\gamma_\nu$$
$$+ b(M^2, M^2 + 2pk, \varkappa^2)\,\sigma_{\nu\rho}\varkappa_\rho$$
$$+ c(M^2, M^2 + 2pk, \varkappa^2)\,\sigma_{\nu\rho}(p' + p - k)_\rho,$$
$$\Gamma^0_\nu(p'+k, p) = a(M^2 - 2p'k, M^2, \varkappa^2)\,\gamma_\nu$$
$$+ b(M^2 - 2p'k, M^2, \varkappa^2)\,\sigma_{\nu\rho}\varkappa_\rho$$
$$+ c(M^2 - 2p'k, M^2, \varkappa^2)\,\sigma_{\nu\rho}(p' + p + k)_\rho. \tag{23}$$

The absence of a pole in $M^{(2)}_{\nu\mu}$ as $k \to 0$ allows one to use (22) for a single-valued determination of $M^{(2)}_{\nu\mu}$ accurate to constant terms. Indeed, expanding $M^{(2)}_{\nu\mu}$ in a series of k and retaining the first term of the expansion, we obtain

$$M^{(2)}_{\nu\mu} = -e\bar{v}(p')\,[ie\partial\Gamma^0_\nu(p', p-k)/\partial k_\mu$$
$$+ ie\partial\Gamma^0_\nu(p'+k, p)/\partial k_\mu]\,|_{k=0}v(p). \tag{24}$$

Retaining the corresponding terms in the expansion of $M^{(1)}_{\nu\mu}$ in k, we find, by means of relation (23) that in the sum $M^{(1)}_{\nu\mu} + M^{(2)}_{\nu\mu}$ the derivatives with respect to the masses drop out, and we finally obtain the following expression for T^{II}_μ:

$$T^{\mathrm{II}}_\mu = -\varkappa^{-2}\,\bar{u}(q')\, ie\gamma_\nu u(q)\,\bar{v}(p')\Big\{\Big[ieF_1(\varkappa^2)\,\gamma_\nu$$
$$- \frac{ie}{2M}\mu_pF_2(\varkappa^2)\,\sigma_{\nu\rho}\varkappa_\rho\Big]\frac{1}{i\gamma(p-k)+M}\Big(ie\gamma_\mu + \frac{ie}{2M}\mu_p\sigma_{\mu\rho}k_\rho\Big)$$
$$+ \Big(ie\gamma_\mu + \frac{ie}{2M}\mu_p\sigma_{\mu\rho}k_\rho\Big)\frac{1}{i\gamma(p'+k)+M}$$
$$\times\Big[ieF_1(\varkappa^2)\,\gamma_\nu - \frac{ie}{2M}\mu_p\sigma_{\nu\rho}\varkappa_\rho F_2(\varkappa^2)\Big]\Big\}\, v(p). \tag{25}$$

In obtaining this formula, we used the above-mentioned properties of the functions a, b, and c. Hence the amplitude for the process of radiation of low energy γ quanta in ep scattering is entirely expressed through the electromagnetic form factors of the proton with an accuracy of terms of the order 1/k and constants. We note that formula (25) agrees with the expression obtained by the dispersion method in the one-nucleon approximation.[6]

CONCLUSION

In conclusion, we derive an expression for the differential cross section for the emission of low energy bremsstrahlung by means of the amplitude we have found [formulas (1), (2), and (25)]. As the independent variables, we choose the energy of the incident electron q_0, the energy of the γ quantum ω, and the following three angles: the angle θ between the directions of the momenta **k** and **q**, the angle θ' between **k** and **q'**, and the angle φ between the directions of the normals to the planes (**k**, **q**) and (**k**, **q'**) (laboratory system). All the remaining variables should be expressed in terms of the independent ones; the cross section should then be expanded in a series in ω.

Since the obtained amplitude is valid only to an accuracy of terms of the order $1/\omega$ and constants, we should retain only the first two terms in this expansion.

Hence

$$d\sigma = d\sigma_0/\omega + d\sigma_1. \tag{26}$$

We obtain the following expression for $d\sigma_0$:

$$d\sigma_0 = \omega d\sigma\Big|_{\omega=0} = \alpha\,(2\pi)^{-2}\omega^2\left[\left(\frac{q'}{q'k}-\frac{q}{qk}\right)-\left(\frac{p'}{p'k}-\frac{p}{pk}\right)\right]^2\Big|_{\omega=0} d\Omega_k d\omega d\sigma_p. \tag{27}$$

Here $d\sigma_p$ is the elastic scattering cross section for electrons of energy q_0 on protons, and $\alpha = e^2/4\pi = 1/137$. The factor in front of $d\sigma_p$ in formula (27) represents the probability of the radiation of a photon when electrons are scattered and goes over, for nonrelativistic particles, into the general expression for the probability of dipole radiation:*

$$\frac{\alpha}{(2\pi)^2}\left\{\frac{1}{m}[\mathbf{q}'-\mathbf{q},\mathbf{n}]-\frac{1}{M}[\mathbf{p}'-\mathbf{p},\mathbf{n}]\right\}^2\frac{d\omega}{\omega}d\Omega_k,$$

where **n** is a unit vector in the direction of flight of the photon.

In the case of ultrarelativistic electrons in which we are interested, we have the following well-known expression for $d\sigma_p$:[7]

$$d\sigma_p = [F_1^2 + (\mu_p F_2/2M)^2\varkappa_1^2 + (\varkappa_1^2/2M^2)\,(F_1+\mu_p F_2)^2\ \mathrm{tg}^2\,(\theta_1/2)]\,d\sigma_0. \tag{28}^{\dagger}$$

Here θ_1 is the angle between **q** and **q'**,

$$d\sigma_0 = \alpha^2\cos^2\frac{\theta_1}{2}\left[4q_0^2\left(1+\frac{2q_0}{M}\sin^2\frac{\theta_1}{2}\right)\sin^4\frac{\theta_1}{2}\right]^{-1}d\Omega_{q'},$$

$$\varkappa_1^2 = \varkappa^2\Big|_{\omega=0} = 4q_0^2\sin^2\frac{\theta_1}{2}\left[1+\frac{2q_0}{M}\sin^2\frac{\theta_1}{2}\right]^{-1}.$$

Expression (27) for $d\sigma_0$ is quite obvious; it is determined by the polar term of the amplitude. The method employed by us actually permits one to find the next term of the expansion $d\sigma_1$. After rather lengthy calculations, we obtain (all quantities are taken for $\omega = 0$)

$$\begin{aligned}
d\sigma_1 = {}& \frac{\alpha^3}{(2\pi)^2}\Big\{A_q^2\varkappa k\left[f_1\left(2M^2+\frac{1}{2}\varkappa^2\right)-2f^2\varkappa^2\right]-\frac{1}{2}A_p^2\varkappa k f_1\varkappa^2\\
&-2A_qA_p(\varkappa k)\,(f_1M^2-f^2\varkappa^2)\\
&+f_1\,(pq+pq')\,(A_q-A_p)\left[\left(\frac{q'}{q'k}+\frac{q}{qk}\right)(p'k+pk)\right.\\
&\left.-\left(\frac{p'}{p'k}+\frac{p}{pk}\right)(qk+q'k)-4\,(p-q')\right]\\
&-2\,(A_q-A_p)^2 f_1\left[(pq)\,(kq')+(pq')\,(kq)+\frac{1}{2}(pk)\,\varkappa^2\right]\\
&-2(\varkappa k)\,(A_q-A_p)\,A_q\left[4\left(2F_1F_1'+2\left(\frac{\mu_p}{2M}\right)^2F_2F_2'\varkappa^2\right.\right.\\
&\left.\left.+\left(\frac{\mu_p}{2M}F_2\right)^2\right)(pq)\,(pq')\cos^2\frac{\theta_1}{2}+ff'\varkappa^4\right]\\
&+(A_q-A_p)^2\left[4\left(2F_1F'_1\varkappa^2\right.\right.\\
&\left.+2\left(\frac{\mu_p}{2M}\right)^2F_2F_2'\varkappa^4+\left(\frac{\mu_p}{2M}F_2\right)^2\varkappa^2+f_1\right)(pq)\,(pq')\cos^2\frac{\theta_1}{2}\\
&\left.+f\varkappa^4\,(f'\varkappa^2+f)\right]\frac{dq_0'}{d\omega}\frac{1}{q_0'}\omega\Big\}\,q_0'\omega\left[\varkappa^4q_0M^2\left(1+\frac{2q_0}{M}\sin^2\frac{\theta_1}{2}\right)\right]^{-1}\Big|_{\omega=0}\\
&\times d\Omega_kd\omega d\Omega_{q'}+\frac{\alpha}{(2\pi)^2}\left[\frac{4\varkappa k}{\varkappa^2}(A_q-A_p)\,A_q\omega+(A_q-A_p)^2\frac{1}{q_0}\right.\\
&\left.\times\left(1+\frac{2q_0}{M}\sin^2\frac{\theta}{2}\right)\omega^2+\frac{2}{p'k}\omega\right]\Big|_{\omega=0}d\omega\,d\Omega_k\,d\sigma_p+\frac{2\alpha}{(2\pi)^2}\\
&\times\left[\left(1+\frac{m^2\omega}{q_0'q'k}\cos\theta'\right)\left(\frac{m^2}{(q'k)^2}-\frac{p'q'}{(p'k)^2}+\frac{(p'q)\,(q'k)}{(p'k)^2\,qk}\right)+\frac{m^2\omega\cos\theta'}{(q'k)^2\,q_0'}\right.\\
&\times\left(\frac{qq'}{qk}+\frac{p'q'}{p'k}-\frac{pq'}{pk}\right)-\frac{M^2q'k}{(p'k)^3}+\frac{pq'}{(pk)\,(p'k)}-\frac{(pp')\,(q'k)}{(pk)\,(p'k)^2}\\
&\left.+\frac{m^2}{(q'k)\,(p'k)}-\frac{qq'}{(p'k)\,(qk)}\right]\frac{dq_0'}{d\omega}\frac{1}{q_0'}\omega^2\Big|_{\omega=0}d\omega d\Omega_kd\sigma_p.
\end{aligned}$$

Here

$$A_q = q'/(q'k)-q/(qk),\qquad A_p = p'/(p'k)-p/(pk),$$

$$f = F_1+\mu_pF_2,\qquad f_1 = F_1^2+(\mu_pF_2/2M)^2\varkappa^2;$$

$$\frac{dq_0'}{d\omega}\frac{1}{q_0'}\Big|_{\omega=0} = 2\left[M\left(1+\frac{2q_0}{M}\sin^2\frac{\theta_1}{2}\right)\right]^{-1}\sin^2\frac{\theta'}{2}-\frac{1}{q_0}\left(1+\frac{2q_0}{M}\sin^2\frac{\theta}{2}\right).$$

$$F_1' = \frac{dF_1}{d\varkappa^2},\quad \text{etc.}$$

Hence the differential cross section for the radiation of γ quanta in ep scattering is fully determined in our approximation by the electromagnetic form factors of the proton F_1 and F_2 and their derivatives with respect to the 4-momentum. Therefore the experimental investigation of the radiation of low energy γ quanta would allow one to obtain additional information on the behavior of these important quantities.

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*$[\mathbf{q}'-\mathbf{q},\mathbf{n}] = (\mathbf{q}'-\mathbf{q}')\times\mathbf{n}$. †tg = tan.

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A FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX IN NORMAL PRODUCTS OF ASYMPTOTIC FIELDS

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Formulas which formally express the coefficient functions of the scattering matrix expansion in normal products of asymptotic fields in terms of time-ordered products of the current operators and a certain set of operators Λ_ν are derived by an axiomatic method without recourse to perturbation theory. Systems of equations for the coefficient functions of the scattering matrix are also established.

1. INTRODUCTION. NOTATION

THE usual approach to quantum field theory based on the Hamiltonian formalism is beset with a number of well-known difficulties, the gravest of which is the impossibility to go beyond the limitations of perturbation theory: we are unable to formulate one of the most essential parts of the theory, namely, the "rules for the removal of divergences," without recourse to an expansion in a small coupling parameter.

This circumstance stimulated attempts to approach the theory "from the other end," as it were. Instead of writing down a Lagrangian or equations of motion and attempting to solve these, one tries to formulate the physical requirements which the solutions have to satisfy and seeks all of those solutions which fulfill these requirements. Bogolyubov[1] and Bogolyubov and Shirkov[2] have used this approach in a consistent manner on the basis of perturbation theory and the hypothesis of the adiabatic switching-on and -off of the interaction. They found results which are in essential agreement with those obtained by the Hamiltonian method.

The new approach, without the simplifications just mentioned, (often, though not too felicitously, called the "axiomatic method") has been intensively developed in recent years in connection with the study of the dispersion relations — the only rigorous result of quantum field theory.

The basic physical requirements of the axiomatic method can be formulated in various ways. For example, we can start from the requirement that there exist Heisenberg field operators at every point which commute on any space-like surface. This is the approach used by Lehmann, Symanzik, and Zimmermann and others (see references 3 and 4 and numerous subsequent papers). On the other hand, one may begin with the program proposed years ago by Heisenberg[5] and restrict oneself to the consideration of the scattering matrix. The latter approach was chosen by Bogolyubov, Polivanov, and the author (reference 6, henceforth referred to as PTDR) in connection with the theory of dispersion relations.

We note at once that it is not quite correct to say that we follow the program of Heisenberg. In fact, we shall deal with a larger set of objects to be studied and an expanded system of basic physical axioms; the class of theories under consideration will be correspondingly narrower. In the Heisenberg program, only those matrix elements of the S matrix are considered which correspond to transitions between stable asymptotic states in which the total energy-momentum is conserved and the squares of all initial and final four-momenta are equal to the corresponding masses; these matrix elements are said to be on the energy shell. The set of matrix elements of this kind can be represented in the form of a functional expansion in the creation and annihilation operators of the type given by (2.14) of PTDR, or in the form of a functional expansion in normal products of asymptotic fields of the type (2.15) of PTDR or (10) below, which satisfy the equation

$$(\Box - m^2)\varphi(x) = 0. \tag{1}$$

However, it is impossible to formulate the strict causality condition if one restricts oneself to the scattering matrix on the energy shell [this can already be seen from the fact that one cannot construct a four-dimensional δ function from the solutions of Eq. (1)]. There will, in this case, be no difference between a theory obeying strict causality and one obeying macroscopic causality. In order to be able to formulate the strict causality condition, we must extend the meaning of the functional expansion (10) and regard it as being an expansion in arbitrary functions $\varphi(x)$ which do not have to satisfy Eq. (1).

Through this remark the method of PTDR comes closer to the methods which presuppose the existence of Heisenberg fields at every point; however, in the last-mentioned methods the class of considered theories is narrowed down further, which, in our opinion, is not warranted by the physics of the problem.

We shall start from the basic assumptions formulated in Sec. 2 of PTDR. For simplicity, we shall deal with a single self-interacting scalar field

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} \{e^{ikx} a^+(\mathbf{k}) + e^{-ikx} a(\mathbf{k})\},$$

$$k^0 = +\sqrt{\mathbf{k}^2 + m^2}, \tag{2}$$

where $a^+(\mathbf{k})$ and $a(\mathbf{k})$ are the creation and annihilation operators for the asymptotic particles (more precisely, the "outgoing" particles), which satisfy the usual commutation relations

$$a(\mathbf{k})\, a^+(\mathbf{k}') - a^+(\mathbf{k}')\, a(\mathbf{k}) = \delta(\mathbf{k} - \mathbf{k}'). \tag{3}$$

We assume that there are no bound states. We can then regard the set of states of the type

$$|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = a^+(\mathbf{k}_1) \ldots a^+(\mathbf{k}_n)\, |0\rangle \tag{4}$$

as forming a complete system as specified by the condition I, (4) of PTDR, i.e., we can assume that

$$\langle\alpha|AB|\beta\rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int ds_1 \ldots ds_s$$

$$\times \langle\alpha|A|\mathbf{s}_1, \ldots, \mathbf{s}_s\rangle \langle\mathbf{s}_1, \ldots, \mathbf{s}_s|B|\beta\rangle \tag{5}$$

for any two operators A and B.

We shall make use of the four-dimensional Fourier transformation in the form

$$F(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} \widetilde{F}(k)\, dk, \quad \widetilde{F}(k) = \int e^{ikx} F(x)\, dx \tag{6}$$

and of the analogous formulas for functions of several arguments. If we define

$$\langle 0|\varphi(x)\,\varphi(y)|0\rangle = -iD^{(-)}(x-y),$$

$$-\langle 0|\varphi(y)\,\varphi(x)|0\rangle = -iD^{(+)}(x-y), \tag{7}$$

we must set, in accordance with (2), (3), and (6),

$$\widetilde{D}^{(-)}(k) = 2\pi i\vartheta(k^0)\,\delta(k^2 - m^2),$$

$$\widetilde{D}^{(+)}(k) = -2\pi i\vartheta(-k^0)\,\delta(k^2 - m^2). \tag{8}$$

Analogously, we have in our normalization

$$[\varphi(x), \varphi(y)] = -iD(x-y),$$

$$\widetilde{D}(k) = 2\pi i\varepsilon(k^0)\,\delta(k^2 - m^2). \tag{9}$$

2. CERTAIN PROPERTIES OF THE SCATTERING MATRIX

The basic quantity in our investigation will be the scattering matrix extended beyond the energy shell. We consider it to be written in the form of a functional expansion in normal products of asymptotic fields:

$$S = \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int dx_1 \ldots dx_\nu \Phi^\nu(x_1, \ldots, x_\nu) :\varphi(x_1) \ldots \varphi(x_\nu):, \tag{10}$$

where the coefficient functions $\Phi^n(x_1, \ldots, x_n)$ are classical functions which are symmetric in all their arguments. We emphasize that $\varphi(x)$ is not restricted by the condition (1).

Variational differentiation of the scattering matrix leads to operators which depend on the points in space-time. Here it is more convenient to use the radiation operators

$$S^n(x_1, \ldots, x_n) = (\delta^n S / \delta\varphi(x_1) \ldots \delta\varphi(x_n))\, S^+ \tag{11}$$

instead of the functional derivatives themselves.

We shall now prove various lemmas which establish a connection between the vacuum expectation values of these radiation operators, the coefficient functions of the scattering matrix, and the matrix elements of the latter.

Lemma 1. The coefficient functions of the scattering matrix coincide (up to a factor) with the vacuum expectation values of the radiation operators (11):

$$\Phi^{(n)}(x_1, \ldots, x_n) = i^n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} S^+ \right| 0 \right\rangle$$

$$= i^n \langle 0|S^{(n)}(x_1, \ldots, x_n)|0\rangle. \tag{12}$$

Proof: We obtain for the n-th variational derivative of the expansion (10)

$$\frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)}$$

$$= \sum_{\nu=n}^{\infty} \frac{(-i)^\nu}{\nu!} \int dz_1 \ldots dz_\nu \Phi^\nu(z_1, \ldots, z_\nu) \frac{\nu!}{(\nu-n)!}$$

$$\times \delta(z_1 - x_1) \ldots \delta(z_n - x_n) :\varphi(z_{n+1}) \ldots \varphi(z_\nu):$$

or, after relabelling the variables,

$$\frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} = (-i)^n \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int dz_1 \ldots dz_\nu \Phi^{n+\nu}$$

$$\times (x_1, \ldots, x_n, z_1, \ldots, z_\nu) :\varphi(z_1) \ldots \varphi(z_\nu):.$$

Only the term containing no normal products on the right-hand side gives a contribution to the vacuum expectation value:

$$i^n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)} \right| 0 \right\rangle = i^n(-i)^n \Phi^n(x_1, \ldots, x_n).$$

To prove the equality (12), we only have to introduce S^+ between the brackets on the left-hand side, which is possible by virtue of the stability of the vacuum, I, (6) of PTDR.

Corollary. The scattering matrix can be written in the form

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)} S^+ \right| 0 \right\rangle \times :\varphi(x_1)\ldots\varphi(x_n):. \tag{13}$$

Lemma 1 expresses the coefficient functions of the scattering matrix in terms of the vacuum expectation values of the operators (11). We now wish to express the matrix elements of the S matrix (on the energy shell) in terms of these same vacuum expectation values. For this purpose, we prove the following lemma:

Lemma 2.

$$\begin{aligned}\langle \mathbf{l}_1, \ldots, \mathbf{l}_n | S | \mathbf{k}_1, \ldots, \mathbf{k}_m \rangle = &\sum_{s=0}^{\min(n,m)} P\left(\frac{l_1, \ldots, l_{n-s}}{l_{n-s+1}, \ldots, l_n}\right) \\ &\times P\left(\frac{k_1, \ldots, k_{m-s}}{k_{m-s+1}, \ldots, k_m}\right) P(k_{m-s+1}, \ldots, k_m) \\ &\times \delta(\mathbf{l}_{n-s+1} - \mathbf{k}_{m-s+1}) \ldots \delta(\mathbf{l}_n - \mathbf{k}_m) \\ &\times \int \frac{dx_1 \ldots dx_{n-s} dy_1 \ldots dy_{m-s} \exp\left\{i\left(\sum_1^{n-s} lx - \sum_1^{m-s} ky\right)\right\}}{(2\pi)^{3(n+m)/2-3s}(2l_1^0 \ldots 2l_{n-s}^0 \cdot 2k_1^0 \ldots 2k_{m-s}^0)^{1/2}} \\ &\times \left\langle 0 \left| \frac{\delta^{n+m-2s} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{n-s})\,\delta\varphi(y_1)\ldots\delta\varphi(y_{m-s})} S^+ \right| 0 \right\rangle, \\ k_j^0 = +\sqrt{\mathbf{k}_j^2 + m^2}, &\qquad l_j^0 = +\sqrt{\mathbf{l}_j^2 + m^2}.\end{aligned} \tag{14}$$

Proof: Consider the matrix element $<\mathbf{l}_1, \ldots, \mathbf{l}_n | S | \mathbf{k}_1, \ldots, \mathbf{k}_m>$, which, according to (4), can be written in the form

$$\langle 0 | a(\mathbf{l}_1) \ldots a(\mathbf{l}_n)\, S\, a^+(\mathbf{k}_1) \ldots a^+(\mathbf{k}_m) | 0 \rangle.$$

If all momenta $\mathbf{k}_i$ and $\mathbf{l}_i$ of the different groups were different, we could immediately commute these operators with the S matrix with the help of (2.20) of PTDR and would obtain the expression (2.21) of PTDR. However, we cannot restrict ourselves to this case, since the coincidence of some $\mathbf{l}$ with some $\mathbf{k}$ gives rise to nonvanishing contributions to integrals over the matrix elements owing to the singularity in the commutation relations (3).

If some $\mathbf{l}$ coincide with some $\mathbf{k}$, additional terms appear: any one of the operators $a^+(\mathbf{k})$ can "combine" with any one of the operators $a(\mathbf{l})$ instead of the S matrix. It is therefore clear that the full expression for the matrix element under consideration must consist of a sum of terms of the type (2.21) of PTDR, where the order of the variational derivative, corresponding to the number of factors of the form $\delta(\mathbf{l} - \mathbf{k})$, decreases by two as we go from one term to the next. Bogolyubov and Shirkov[2] have introduced the symmetrization operator $P(x_1, \ldots, x_s | x_{s+1}, \ldots, x_n)$, which denotes the sum over all possible $n!/[s!(n-s)!]$ decompositions of the set of arguments $\{x_1, \ldots, x_n\}$ into two groups of s and n − s arguments, where decompositions which differ from each other only by a permutation within the group are counted only once, and the operator $P(x_1, \ldots, x_r)$, which denotes the sum over all r! permutations within the group $\{x_1, \ldots, x_r\}$. It does not require much imagination to see that, with the help of these operators, the sum under consideration reduces* precisely to the form on the right-hand side of (14) (the factor S^+ can be introduced between the brackets of the vacuum expectation value by virtue of the stability of the vacuum).

The upper limit in the sum over s on the right-hand side of (14) corresponds to the "exhaustion of the supply" of creation and annihilation operators; the first sum over the permutations corresponds to all possible decompositions of the set of annihilation operators into one group which combines with the creation operators and another group which combines with the S matrix; the second sum over the permutations corresponds to the same decompositions of the set of creation operators; finally, the last summation over $P(k_{m-s+1}, \ldots, k_m)$ goes over all possible different pairings of the operators $a(\mathbf{l})$ and $a^+(\mathbf{k})$ combining with one another.

It is clear that a formula completely analogous to Lemma 2 will be valid for the matrix elements of variational derivatives of the S matrix of any order. The additional derivatives can be "pulled" inside" the brackets of the vacuum expectation values without change of form. This is not the case for the radiation operators (11). The presence of the factor S^+ requires an additional expansion in terms of a complete set of functions, and the formulas become, in general, much more complicated. The only exception to this are those cases in which the right-hand bracket contains the vacuum or a single-particle state. By virtue of the stability condition I, (6) of PTDR, we can then "pull the factor S⁺ inside," and the derivation is the same as for Lemma 2.

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*Cf. the technique developed in reference 7; with the help of this technique the discussion above could be replaced by calculations (which, admittedly, are very involved).

Thus we can formulate two more lemmas:

Lemma 3.

$$\left\langle \mathbf{l}_1, \ldots, \mathbf{l}_n \left| \frac{\delta^r S}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)} \cdot S^+ \right| 0 \right\rangle$$

$$= \int \frac{dx_1 \ldots dx_n \exp\left\{ i \sum_1^n lx \right\}}{(2\pi)^{3n/2} \sqrt{2l_1^0} \ldots \sqrt{2l_n^0}}$$

$$\times \left\langle 0 \left| \frac{\delta^{r+n} S}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)\,\delta\varphi(x_1)\ldots\delta\varphi(x_n)} \cdot S^+ \right| 0 \right\rangle. \qquad (15)$$

By complex conjugation we obtain the

Corollary

$$\left\langle 0 \left| S \frac{\delta^r S^+}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)} \right| \mathbf{k}_1, \ldots, \mathbf{k}_m \right\rangle$$

$$= \int \frac{dy_1 \ldots dy_m \exp\left\{ -i \sum_1^m ky \right\}}{(2\pi)^{3m/2} \sqrt{2k_1^0} \ldots \sqrt{2k_m^0}}$$

$$\times \left\langle 0 \left| S \frac{\delta^{r+m} S^+}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)\,\delta\varphi(y_1)\ldots\delta\varphi(y_m)} \right| 0 \right\rangle. \qquad (16)$$

Lemma 4.

$$\langle \mathbf{l}_1, \ldots, \mathbf{l}_n \,|\, S^{(r)}(z_1, \ldots, z_r) \,|\, \mathbf{k} \rangle$$

$$= (-i)^{r+n+1} \int \frac{dx_1 \ldots dx_n dy \exp\left\{ i \sum_1^n lx - iky \right\}}{(2\pi)^{3(n+1)/2} \sqrt{2l_1^0 \ldots 2l_n^0 2k^0}}$$

$$\times \Phi^{r+n+1}(z_1, \ldots z_r, x_1, \ldots, x_n, y) + P\left(\frac{l_1, \ldots, l_{n-1}}{l_n} \right)$$

$$\times (-i)^{r+n-1} \int \frac{dx_1 \ldots dx_{n-1}}{(2\pi)^{3(n-1)/2} \sqrt{2l_1^0, \ldots 2l_{n-1}^0}} \exp\left\{ i \sum_1^{n-1} lx \right\}$$

$$\times \Phi^{r+n-1}(z_1, \ldots, z_r, x_1, \ldots, x_{n-1})\, \delta(\mathbf{l}_n - \mathbf{k}). \qquad (17)$$

Complex conjugation leads again to a formula of the type (16).

Up to this point we have not made any use of the causality condition. In the system of basic assumptions of PTDR, this condition had the form

$$\frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y)} S^+ \right) = \frac{\delta S^{(1)}(y)}{\delta\varphi(x)} = 0 \text{ for } x \lesssim y. \qquad (18)$$

[The sign of the inequality corresponds to the fact that we regard the asymptotic fields $\varphi(x)$ as "outgoing" fields; the inequality would be the opposite for "incoming" fields.] We now prove several new relations which follow from this condition.

Lemma 5. The causality condition (18) implies that

$$\frac{\delta}{\delta\varphi(x)} S^n(y_1, \ldots, y_n) = 0, \qquad (19)$$

if

$$x \lesssim \{y_1, \ldots, y_n\}. \qquad (19a)$$

[The inequality (19a) means that the point x is earlier than, or lies on a space-like surface with respect to, all points $y_1, \ldots, y_n$.]

Proof: Assume that the lemma is valid up to a certain value N. Then

$$\frac{\delta}{\delta\varphi(x)} \left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} \cdot S^+ \right) = 0, \quad \text{if} \quad x \lesssim \{y_1, \ldots, y_N\}. \qquad (19N)$$

Differentiating this equation with respect to $\varphi(y_{N+1})$, we find

$$\frac{\delta}{\delta\varphi(x)} \left(\frac{\delta^{N+1} S}{\delta\varphi(y_1)\ldots\delta\varphi(y_{N+1})} S^+ \right)$$

$$+ \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+ S \frac{\delta S^+}{\delta\varphi(y_{N+1})} \right) = 0,$$

using the unitarity condition I, (5) of PTDR and applying the sum rule for the derivative with respect to $\varphi(x)$, we obtain

$$\frac{\delta}{\delta\varphi(x)} \left(\frac{\delta^{N+1} S}{\delta\varphi(y_1)\ldots\delta\varphi(y_{N+1})} S^+ \right)$$

$$= \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+ \right) \cdot \frac{\delta S}{\delta\varphi(y_{N+1})} S^+$$

$$+ \frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+ \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y_{N+1})} S^+ \right).$$

The first term vanishes because of (19N) and the second term also does by virtue of (18), if we require that $x \lesssim y_{N+1}$. The validity of (19N) for n = N+1 therefore follows from the validity of (19N) for n = N. Since (19N) holds for n = 1, the lemma is proved.

Lemma 5 allows us to increase arbitrarily the number of internal arguments in the causality condition (18). The number of external arguments can also be increased. The corresponding lemma is

Lemma 6. The causality condition implies

$$\frac{\delta^m}{\delta\varphi(x_1)\ldots\delta\varphi(x_m)} S^{(n)}(y_1, \ldots, y_n) = 0, \qquad (20)$$

if for at least one $1 \leq j \leq m$

$$x_j \lesssim \{y_1, \ldots, y_m\}. \qquad (20a)$$

The proof is obvious.

The original form of the causality condition (18), as well as the subsequent relations (19) and (20), contains not only the radiation operators (11), but also their variational derivatives. Under certain conditions it is more convenient to use a form of the causality condition which involves only the operators (11) themselves, in analogy to the "integral'' causality condition in perturbation theory.[8] This form is established by the following lemma:

Lemma 7. The causality condition (18) implies that the radiation operators $S^{(n)}(x_1, \ldots, x_n)$ can be represented in the form

$$S^{(n)}(x_1, \ldots, x_n) = S^{(s)}(x_{j_1}, \ldots, x_{j_s})\, S^{(n-s)}(x_{j_{s+1}}, \ldots, x_{j_n}), \tag{21}$$

if

$$\{x_{j_1}, \ldots, x_{j_s}\} \gtrsim \{x_{j_{s+1}}, \ldots, x_{j_n}\} \tag{21a}$$

where $0 \leq s \leq n$.

Proof: Assume that the lemma is valid for $n \leq N$ and that the arguments $\{x_1, \ldots, x_{N+1}\}$ have the property

$$\{x_{j_1}, \ldots, x_{j_s}\} \gtrsim \{x_{j_{s+1}}, \ldots, x_{j_{N+1}}\}. \tag{21b}$$

Expressing, as usual, $S^{(k)}$ in terms of variational derivatives, we have the identity

$$\frac{\delta^{N+1} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{N+1})} S^+ = \frac{\delta}{\delta\varphi(x_{j_{N+1}})}\left(\frac{\delta^N S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})\,\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+\right) - \frac{\delta^N S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+ S\, \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})}.$$

According to our assumption, Lemma 7 is valid for the radiation operators of the N-th order; we can therefore continue the equation above by applying the lemma and using the sum rule for the differentiation with respect to $\varphi(x_{j_{N+1}})$ in the first term:

$$\begin{aligned}\frac{\delta^{N+1} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{N+1})} S^+ &= \frac{\delta}{\delta\varphi(x_{j_{N+1}})}\left(\frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+\right) \frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+ \\ &+ \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s+1} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_{N+1}})} S^+ \\ &+ \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})} \\ &- \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})}.\end{aligned}$$

The first term vanishes because of Lemma 5, since the point $x_{j_{N+1}}$ belongs to the second group in (21b); the next two terms cancel each other. Returning to the operators $S^{(k)}$, we see that the lemma is also valid for $n = N+1$. Since it is trivial for $n = 1$, the lemma is proved by induction.

The integral causality condition (21) is not only necessary, but also sufficient for the validity of the differential condition (18). It turns out that the latter can be proved if (21) is satisfied only for $n = 2$. Indeed, let us use the sum rule for the variational derivative with respect to $\varphi(x)$ on the left-hand side of (18):

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = \frac{\delta^2 S}{\delta\varphi(x)\,\delta\varphi(y)} S^+ + \frac{\delta S}{\delta\varphi(y)} S^+ S \frac{\delta S^+}{\delta\varphi(x)}.$$

According to (21), the first term is equal to

$$\frac{\delta S}{\delta\varphi(y)} S^+ \frac{\delta S}{\delta\varphi(x)} S^+ \text{ for } x \lesssim y,$$

$$\frac{\delta S}{\delta\varphi(x)} S^+ \frac{\delta S}{\delta\varphi(y)} S^+ \text{ for } y \lesssim x,$$

and the second term always gives, owing to the unitarity condition,

$$-\frac{\delta S}{\delta\varphi(y)} S^+ \frac{\delta S}{\delta\varphi(x)} S^+.$$

Thus we find that (21) implies

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = \begin{cases}\left[\dfrac{\delta S}{\delta\varphi(x)} S^+,\ \dfrac{\delta S}{\delta\varphi(y)} S^+\right]_-, & x \gtrsim y; \\ 0 & x \lesssim y.\end{cases} \tag{22}$$

The following lemma is proved in the same way:

Lemma 8. For the validity of the differential causality condition it is sufficient that the integral causality condition hold in the form

$$S^{(2)}(x_1, x_2) = S^{(1)}(x_{j_1}) S^{(1)}(x_{j_2}), \quad \text{if} \quad x_{j_1} \gtrsim x_{j_2}. \tag{23}$$

Why is condition (23) sufficient for the derivation of (18), whereas, in perturbation theory, we must require[8] the validity of the integral condition for all n in order to obtain the differential causality condition? The reason for this is that our radiation operators $S^{(k)}$ depend not only explicitly on the arguments $x_1, \ldots, x_k$, but also functionally on $\varphi(x)$; this latter dependence relates the $S^{(k)}$ with different indices to each other.

The results of the preceding investigation of the causality condition can be formulated in the form of the following theorem:

Theorem. The forms (18) and (23) of the causality condition are equivalent. Either one of them leads to a system relating the operators $S^{(k)}$ to each other which is expressed by the formulas (19), (20), and (21).

3. SYSTEMS OF EQUATIONS FOR THE COEFFICIENT FUNCTIONS

The various forms of the causality condition obtained in the previous section are operator equations. In many cases it is more convenient to deal with ordinary functions (coefficient functions or matrix elements) instead of operators. The above-mentioned relations can be used to derive various systems of equations for these functions.

For this purpose we note that the causality condition in the form (21) can be written as (disregarding the possible coincidences of the arguments, see below)

$$\frac{\delta^n S}{\delta\varphi(z_1)\ldots\delta\varphi(z_n)}S^+$$

$$= P\left(\frac{z_1,\ldots,z_s}{z_{s+1},\ldots,z_n}\right)\Theta(z_1,\ldots,z_s;\ z_{s+1},\ldots,z_n)$$

$$\times\frac{\delta^s S}{\delta\varphi(z_1)\ldots\delta\varphi(z_s)}S^+\frac{\delta^{n-s}S}{\delta\varphi(z_{s+1})\ldots\delta\varphi(z_n)}S^+, \tag{24}$$

where $\Theta(z_1,\ldots,z_s;z_{s+1},\ldots,z_n)$ equals unity if all $z_1^0,\ldots,z_s^0$ are greater than all $z_{s+1}^0,\ldots,z_n^0$, and zero otherwise. If we take the vacuum expectation value of this equation, we obtain on the left the function $(-i)^n\Phi^n(z_1,\ldots,z_n)$, according to Lemma 1. The right-hand side must be expanded in a complete set of states with the help of (4). Here we are faced with two possibilities.

If only one argument belongs to the first group in (24), $s = 1$, we can use the unitarity condition to write the first factor on the right-hand side in the form

$$\frac{\delta S}{\delta\varphi(z_1)}S^+ = -S\frac{\delta S^+}{\delta\varphi(z_1)},$$

it can then be expressed in terms of the functions $\Phi^{\nu *}(z_1,\ldots,z_\nu)$ with the help of the corollary of Lemma 3. The second factor has precisely the form that reduces to the coefficient functions Φ^ν with the help of Lemma 3, and we find

$$\Phi^n(z_1,\ldots,z_n) = P\left(\frac{z_1}{z_2,\ldots,z_n}\right)\Theta(z_1;z_2,\ldots,z_n)$$

$$\times\sum_{m=0}^{\infty}\frac{1}{m!}\int dx_1\ldots dx_m dy_1\ldots dy_m$$

$$\times\int\frac{dk_1\ldots dk_m}{(2\pi)^{3m}2k_1^0\ldots 2k_m^0}\exp\left\{-i\sum_1^m k(y-x)\right\}$$

$$\times\Phi^{m+1*}(z_1,\ y_1,\ldots,y_m)$$

$$\times\Phi^{m+n-1}(x_1,\ldots,x_m,\ z_2,\ldots,z_n).$$

We note that the integrals over each individual k go over into the functions $D^{(-)}(y-x)$,

$$\frac{1}{(2\pi)^3}\int dk_j\frac{\exp\{-ik_j(y_j-x_j)\}}{2k_j^0} = -iD^{(-)}(y_j-x_j),$$

owing to the normalization (6) and (8). We thus obtain the following infinite system of equations for the functions Φ^ν:

$$\Phi^n(z_1,\ldots,z_n) = P\left(\frac{z_1}{z_2,\ldots,z_n}\right)\Theta(z_1;z_2,\ldots,z_n)\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}$$

$$\times\int dx_1,\ldots dx_m dy_1\ldots dy_m\Phi^{1+m*}(z_1,\ y_1,\ldots,y_m)$$

$$\times D^{(-)}(y_1-x_1)\ldots D^{(-)}(y_m-x_m)$$

$$\times\Phi^{n-1+m}(x_1,\ldots,x_m,\ z_2,\ldots,z_n). \tag{25}$$

This system of equations is analogous to the "system A" of Lehmann, Symanzik, and Zimmermann,[4] which was derived from different basic assumptions.

A different possibility, or more precisely, a different series of possibilities arises if $1 < s < n-1$ in (24). In this case the unitarity condition does not any more allow us to bring the first factor on the right-hand side of (24) into a form to which the corollary of Lemma 3 can be applied. In order to express the right-hand side in terms of the functions Φ^ν, we must split it up into three factors, i.e., we must expand twice in a complete system of states. We then obtain the system

$$\Phi^n(x_1,\ldots,x_n) = P\left(\frac{x_1,\ldots,x_s}{x_{s+1},\ldots,x_n}\right)$$

$$\times\Theta(x_1,\ldots,x_s;x_{s+1},\ldots,x_n)\sum_{r=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{{}^{r+\nu+\mu}}{r!\nu!\mu!}$$

$$\times\int dz_1\ldots dz_r du_1\ldots du_\nu\int du_1'\ldots du_\nu' dv_1\ldots dv_\mu\int dz_1'\ldots dz_r' dv_1'\ldots dv_\mu'$$

$$\times\Phi^{s+r+\nu}(x_1,\ldots,x_s,z_1,\ldots,z_r,u_1,\ldots,u_\nu)$$

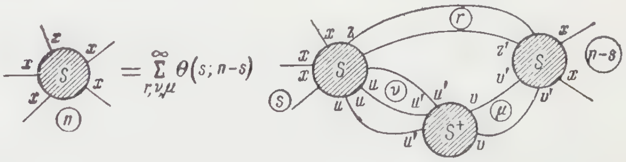
$$\times D^{(-)}(z_1-z_1')\ldots D^{(-)}(z_r-z_r')D^{(-)}(u_1-u_1')\ldots$$

$$\ldots D^{(-)}(u_\nu-u_\nu')\Phi^{\nu+\mu*}(u_1',\ldots,u_\nu',v_1,\ldots,v_\mu)$$

$$\times D^{(-)}(v_1-v_1')\ldots D\quad(v_\mu')$$

$$\times\Phi^{n-s+r+\mu}(z_1',\ldots,z_r',v_1',\ldots,v_\mu',x_{s+1},\ldots,x_n) \tag{26}$$

(more precisely, this is a set of systems corresponding to different values of s). The structure of this system of equations is conveniently visualized with the help of the following graphs:



The infinite system of coupled equations (25) is considerably simpler than the system (26). However, the system (25) alone is not sufficient to determine the functions Φ^ν. This is due to the following very important circumstance. The different forms (18) — (23) of the causality condition derived in the previous section, are valid only if the coordinates are really unequal; for example, the relation (23) is valid only for $x_1 \lesssim x_2$ or $x_2 \lesssim x_1$, but not for $x_1 = x_2$. Thus the expressions (21) or (23) do not determine the functions $\Phi^{\nu}(x_1, \ldots, x_{\nu})$ for all values of the arguments. If any two points belonging to different groups in condition (21a) [or the two points in condition (23)] coincide, the corresponding value of Φ^{ν} is indeterminate. We must, therefore, introduce additional terms ("counterterms") on the right-hand sides of (25) or (26), which correspond to such coincidences of the arguments.*

To include these counterterms in a consistent manner would evidently lead to enormous combinatorial difficulties. However, if we agree to use the system (26) for all values of s simultaneously, we readily observe that the set of points $x_1, \ldots, x_n$ can always be divided up into two groups according to (21a), except when all points coincide. In other words, we only have to introduce one counterterm for each function Φ^n; the other counterterms are automatically included with the help of one of the formulas (26). However, this involves a peculiar kind of "equilibrium," namely, the determination of one and the same function by different equations, depending on the values of the arguments.

The condition (23), which is sufficient for the validity of the causality principle, leads to yet another system of equations. The main advantage of this system is that it contains only one ϑ function; one may therefore hope that the combinatorial difficulties disappear. However, to obtain a closed system, we cannot restrict ourselves to the vacuum expectation value of the operator condition, but must consider matrix elements between all states. Such a system will be discussed by Polivanov and the author in a different place, so that we shall not write it down here.

4. FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX

In the previous section we established several infinite systems of coupled equations for the determination of the coefficient functions Φ^{ν} in the functional expansion (10) of the scattering matrix in normal products of asymptotic fields. The solution of these systems (even approximately) is very difficult. It turns out that we can obtain some information on the functional expansion (10) by proceeding in a different way. Indeed, we shall show that all coefficient functions can be expressed in terms of the vacuum expectation values of T products of a certain set of operators. Here we note the remarkable fact that the structure of the functional expansion (10) is very reminiscent of the structure of the functional expansion in powers of the "interaction inclusion function," or, roughly speaking, of the usual perturbation expansion.

Let us return now to the radiation operators $S^{(n)}$ [formula (11)]. The causality principle is expressed in terms of these operators [formula (21)]. On the other hand, if we perform n variational differentiations on the unitarity condition I, (5) of PTDR, we obtain the following condition for the operators $S^{(n)}$

$$\sum_{m=0}^{n} P\left(\begin{matrix} x_1, \ldots, x_{n-m} \\ x_{n-m+1}, \ldots, x_n \end{matrix}\right) S^{(n-m)}(x_1, \ldots, x_{n-m}) \times S^{(m)+}(x_{n-m+1}, \ldots, x_n) = \delta_{n0}. \tag{27}$$

We can now forget about the original definition (11) of the radiation operators $S^{(n)}$ and consider the purely algebraic problem of finding the general form of the operators which satisfy the conditions (21) and (27). But the conditions (21) and (27) are precisely the conditions which have to be satisfied by the operator coefficient functions in the expansion of the S matrix in powers of the "interaction inclusion function" $g(x)$ [see reference 2, formula (18.1), and reference 8, formula (1)]. Indeed, the condition (21) is algebraically identical to the "integral causality condition" (7) of reference 8, while the condition (27) is the same as the unitarity condition (18.9) of reference 2.

It thus appears that the problem of finding the general form of the radiation operators $S^{(n)}$ is identical with the problem of finding the general form of the coefficient functions of the scattering matrix in perturbation theory. The latter problem has been solved in reference 2 and also in reference 8, where a form of the causality condition analogous to (21) has been used. It was shown that all conditions (21) and (27) are consistent with each other and that each operator function $S^{(n)}(x_1, \ldots, x_n)$ is expressible in terms of the operator functions with lower indices except for the antihermitian part of its value when all arguments are the same. These latter values can be given arbitrarily.

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*Formal calculations with the systems (25) and (26) without taking into account this circumstance lead to the known divergences. The divergences are due to products of ϑ functions which are not sufficiently regular at zero; as was shown in PTDR, Secs. 4 and 5, this is equivalent to an illegitimate application of the Cauchy integral formula to Fourier transforms which do not vanish at infinity.

In our case the general expression for $S^{(n)}$ will therefore have the form

$$S^{(n)}(x_1, \ldots, x_n) = (-i)^n T(\Lambda_1(x_1) \ldots \Lambda_1(x_n))$$
$$+ \sum \frac{(-i)^m}{m!} P(x_1, \ldots, x_{\nu_1} \mid x_{\nu_1+1}, \ldots, x_{\nu_1+\nu_2} \mid$$
$$\ldots \mid x_{\nu_1+\ldots+\nu_{m-1}+1}, \ldots, x_n)$$
$$\times T[\Lambda_{\nu_1}(x_1, \ldots, x_{\nu_1}) \ldots \Lambda_{\nu_m}(x_{\nu_1+\ldots+\nu_{m-1}+1}, \ldots, x_n)]$$
$$- i\Lambda_n(x_1, \ldots, x_n), \qquad (28)$$

where the summation goes over all values of ν which satisfy the inequalities $2 \leq m \leq n-1$, $\nu_1 + \ldots + \nu_m = n$, $\nu_j \geq 1$, and $P(x_1, \ldots, x_{\nu_1} \mid \ldots \mid \ldots, x_n)$ is the operator introduced in reference 2 which implies summation over all possible $n!/[\nu_1!\nu_2!\ldots\nu_m!]$ decompositions of the set of n points into m groups containing $\nu_1, \nu_2, \ldots, \nu_m$ points each.

The operators $\Lambda_\nu(x_1, \ldots, x_\nu)$ are arbitrary operators with the following properties:

1) locality:

$$\Lambda_\nu(x_1, \ldots, x_\nu) = 0, \text{ except when } x_1 = x_2 = \ldots = x_\nu, \qquad (29)$$

2) hermiticity:

$$\Lambda_\nu^+(x_1, \ldots, x_\nu) = \Lambda_\nu(x_1, \ldots, x_\nu), \qquad (30)$$

3) symmetry:

$$\Lambda_\nu(x_{\alpha_1}, \ldots, x_{\alpha_\nu}) = \Lambda_\nu(x_1, \ldots, x_\nu), \qquad (31)$$

4) commutativity for points at space-like distances:

$$[\Lambda_\nu(x_1, \ldots, x_\nu), \Lambda_\mu(y_1, \ldots, y_\mu)]_- = 0, \qquad (32)$$

when

$$(x_1 = \ldots = x_\nu) \sim (y_1 = \ldots = y_\mu).$$

These operators reflect the above-noted freedom in the choice of the values of the radiation operators $S^{(\nu)}$ when all arguments coincide. The first of them is the current operator:

$$\Lambda_1(x) = j(x) = iS^{(1)}(x). \qquad (33)$$

We emphasize that in writing down formula (28) we imagine that for each ν we have somehow fixed the arbitrariness in the definition of the T product for coinciding arguments (rules of integration near zero) in such a way that unitarity is not violated, and then add the operator Λ_ν. We can say, therefore, that the ambiguities of the T products are transferred to the operators Λ_ν; they play the role of the above-mentioned counterterms in the infinite systems of equations. If we do not wish to separate out the counterterms, we can restrict ourselves to the T product of the currents in formula (28); however, whenever there is a coincidence of the arguments, we get an ambiguous expression.*

Since, according to Lemma 1, the coefficient functions Φ^n of the expansion (10) are obtained by taking the vacuum expectation value of the radiation operators $S^{(n)}$, they can also be expressed in terms of the set of operators $\Lambda_1, \ldots, \Lambda_\nu, \ldots$ with the help of formula (28):

$$\Phi^n(x_1, \ldots, x_n) = \langle 0 \mid T[j(x_1) \ldots j(x_n)] \mid 0 \rangle$$
$$+ \sum \frac{i^{n-m}}{m!} P(x_1, \ldots x_{\nu_1} \mid \ldots \mid \ldots, x_n)$$
$$\times \langle 0 \mid T[\Lambda_{\nu_1}(x_1, \ldots, x_{\nu_1}) \ldots \Lambda_{\nu_m}(\ldots, x_n)] 0 \rangle$$
$$+ i^{n-1} \langle 0 \mid \Lambda_n(x_1, \ldots, x_n) \mid 0 \rangle, \qquad (34)$$

where the summation goes over the same values of ν as in formula (28).

5. DISCUSSION

The expressions (28) and (34) for the radiation operators and the coefficient functions of the S matrix are very useful in the elucidation of the structure of the functional expansion (10) of the scattering matrix. In particular, they may turn out to be powerful tools in attempts of solving the infinite system of equations for the functions Φ^ν approximately. On the other hand, they do not, of course, in any way represent a solution of the basic problem of finding the scattering matrix without recourse to perturbation theory.

Indeed, even in the perturbation theory the formula analogous to (28) leads only to a formal construction of the scattering matrix. The solution is formal, because applying it directly leads to divergent expressions when the arguments of the T products coincide. As is well known, the next step is then to impose on the sum analogous to the sum (28) the condition that it contain no divergences; it can be shown (see reference 2, Sec. 26) that it is always possible to satisfy this condition by an appropriate choice of the operators Λ_ν. This step has yet to be taken in our case.

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*We note that in our method the counterterms play a more important role than in perturbation theory. In the latter, they came into play only in problems having to do with infinities and renormalization; on the other hand, it is easily seen that in our method, using the perturbation theoretical expression for the current, the operators Λ_ν with $\nu > 1$ give also a contribution to processes which do not involve divergences, as for example, the Compton effect in lowest order. The reason for this is that the "fundamental" term in (28) or (34) is the T product of the currents and not of the Lagrangians.

The main difference between our formulas and the perturbation theoretical expansion consists, however, in something else. In perturbation theory, the quasilocal operators Λ_ν depend only on the coordinates $x_1, \ldots, x_\nu$ (more precisely, on the free field operators at these coinciding points). In our case the operators Λ_ν depend not only on the coordinates $x_1, \ldots, x_\nu$, but also functionally on $\varphi(y)$:

$$\Lambda_\nu(x_1, \ldots, x_\nu) = \Lambda_\nu(x_1, \ldots, x_\nu \,|\, \varphi(y)). \qquad (35)$$

The same is true, of course, for the operators $S^{(n)}$. So far we do not know any details of this dependence; it is only clear that it will couple operators with different indices to one another.

Taking this circumstance into account, we see that formula (28) expresses the operators $S^{(n)}$ for a certain "fixed" $\varphi(y)$ in terms of products of the operators Λ_ν referring to the same "value" of $\varphi(y)$, and hence in terms of the $\Lambda_\nu(\ldots|\varphi(y))$ with arbitrary $\varphi(y)$ (sum over the complete system!). In order to be able to use formulas (28) and (34) as the basis for the determination of the scattering matrix without recourse to perturbation theory, we must therefore first investigate the character of the functional dependence of the operators Λ_ν on $\varphi(y)$. This problem is the subject of a separate study, which will be presented in a different place.

In conclusion I wish to thank N. N. Bogolyubov and M. K. Polivanov for their continuous interest in this work and for a valuable discussion.

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EQUATIONS FOR PHOTOPRODUCTION OF PIONS ON NUCLEONS WITH EFFECTS DUE TO A PION-PION INTERACTION TAKEN INTO ACCOUNT

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Equations for the partial wave amplitudes for photoproduction of pions on nucleons at low energies are derived from the Mandelstam representation[11] in the Cini-Fubini approximation.[12] Nucleon recoil and pion-pion interaction are taken into account. The latter contributes only to the isotopic scalar photoproduction amplitudes. Expressions for these amplitudes, with effects of pion-pion interaction included, are obtained.

1. INTRODUCTION

RECENTLY there has been considerable interest in the study of pion-nucleon scattering including the effects due to a pion-pion interaction.[1-4] In this paper we consider the same problem for photoproduction.

Calculations based on one dimensional dispersion relations[5-7] give in general satisfactory agreement with present day experiments on photoproduction in the low energy region.[7-9] The discrepancies[9,10] may be thought of as due to an approximate treatment of the nucleon recoil[5-7] and neglect of the pion-pion interaction and of high energy contributions. In this work we obtain from the Mandelstam representation[11] in the Cini-Fubini approximation[12] a set of equations in which nucleon recoil and the pion-pion interaction are taken into account.

The pion-pion interaction enters into the equations under discussion through the amplitude for photoproduction of pions on pions. An expression for this amplitude has been obtained by one of the authors.[13] The pion-pion interaction contributes only to the isotopic scalar photoproduction amplitudes. Consequently the pion-pion resonance in the $J = I = 1$ state (if it exists) contributes only to those photoproduction amplitudes to which the pion-nucleon resonance (in the $J = I = 3/2$ state) does not.

The amplitude for photoproduction on pions depends on high-energy singularities and involves a parameter which depends on the amplitudes for the processes $\gamma\pi \to N\bar{N}$ and $\pi\pi \to N\bar{N}$ in the observable region. Consequently it is possible to write formally a system of coupled equations, which relate the amplitudes for the processes $\gamma N \to \pi N$ and $\gamma\pi \to N\bar{N}$ to the pion-nucleon and pion-pion amplitudes and contain no new parameters. At this time, however, we do not know how to take into account the contributions from the high energy regions. On the other hand, calculations of the amplitude for $\gamma\pi \to N\bar{N}$ in the observable region cannot be trusted if they utilize only the low energy singularities. For that reason it is necessary at this time to use experimental data for the process $\gamma\pi \to N\bar{N}$ (as well as $\pi\pi \to N\bar{N}$). Thus we shall be discussing only the amplitude for the process $\gamma N \to \pi N$, which involves a parameter determined by the process $\gamma\pi \to \pi\pi$ and assumed here to be known experimentally.

In order to write the above-mentioned equations it is necessary to discuss the cumbersome kinematics and unitarity condition for the processes $\gamma N \to \pi N$ and $\gamma\pi \to N\bar{N}$ (Sec. 2). The kinematics of the first process has been discussed by a number of authors,[5,6] the second process (as well as $\pi\pi \to N\bar{N}$) is best discussed using the formalism of Jacob and Wick.[14]

The Mandelstam representation[11] in the Cini-Fubini approximation[12] is written out in Sec. 3 (see also Bowcock et al.[1]). We assume the representation to be valid with no subtractions. One is encouraged in such an approach by the observation[13] that if the differential cross sections for photoproduction and forward scattering of pions on nucleons increase in the same manner at infinity then one needs one less subtraction for photoproduction than for scattering. This is a consequence of gauge invariance which leads to the appearance of an energy dependent factor in the relation between the invariant amplitude (for which one writes the Mandelstam representation) and the photoproduction matrix element, where no such factor appears in the scattering problem.

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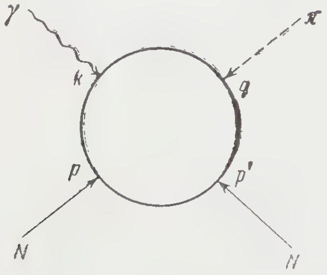
By making use of the kinematic relations of Sec. 2 the integral equations for the partial amplitudes are derived from the Cini-Fubini representation by simple integrations. Here the nucleon recoil is important only in the terms involving the direct pion-photon interaction and the magnetic dipole transitions in the $I = 3/2$ state that do not depend on the pion-pion interaction. This interaction contributes to the isotopic scalar amplitudes, and for those the nucleon recoil may be taken into account approximately. The expressions derived for the isotopic scalar amplitudes contain the effects of a pion-pion interaction and are a generalization of previously obtained results.[5-7]

2. KINEMATICS AND UNITARITY CONDITION

The matrix elements for the processes

$$\begin{aligned} &\mathrm{I.}\ \gamma + N \to \pi + N', \\ &\mathrm{II.}\ \pi + N \to \gamma + N', \\ &\mathrm{III.}\ \gamma + \pi \to N + \bar{N}, \end{aligned}$$

described by the diagram shown (where k stands for the photon momentum, q for the pion momentum, and p and p′ for the momenta of the initial



and final nucleons in reactions I and II or the nucleon and antinucleon in reaction III) are given by

$$S_{fi} = -i\,(2\pi)^4 \delta\,(k + q + p + p')\, m\,(4k^0 q^0 p^0 p'^0)^{-1/2} F, \quad (1)$$

$$F = \left\langle f \left| \left(\delta_{\rho 3} F^{(1)} + \tau_\rho F^{(2)} + \tfrac{1}{2} [\tau_\rho,\ \tau_3]\, F^{(3)} \right) \right| i \right\rangle, \quad (2)$$

$$F^{(\alpha)} = \sum_{i=1}^{4} R_i H_i^{(\alpha)} (s, \bar{s}, t); \quad (3)$$

$$\begin{aligned} R_1 &= \bar{u}\gamma^5 [(pk)\,(p'e) - (pe)\,(p'k)]\, u, \\ R_2 &= \bar{u}\gamma^5 [((p' - p)\, k)\,(\gamma e) - ((p' - p)\, e)\,(\gamma k)]\, u, \\ R_3 &= \bar{u}\gamma^5 [(qk)\,(\gamma e) - (qe)\,(\gamma k)]\, u, \\ R_4 &= \bar{u}\gamma^5 [(\gamma k)\,(\gamma e) - (\gamma e)\,(\gamma k)] u. \end{aligned} \quad (4)$$

Here $|i\rangle$, $|f\rangle$ stand for the isotopic functions of the initial and final states, $\bar{u}$ is the final nucleon spinor, u is the initial nucleon spinor (processes I and II) or the final antinucleon spinor (process III) and e is the photon polarization vector.* The twelve invariant functions $H_i^{(\alpha)}$, which characterize the processes under discussion, depend on the arguments

$$s = (k + p)^2, \quad \bar{s} = (q + p)^2, \quad t = (p + p')^2; \qquad s + \bar{s} + t = 2m^2 + \mu^2 \quad (5)$$

(m is the nucleon mass, μ is the pion mass). From crossing symmetry of F (for processes I and II), or from invariance under charge conjugation (for process III), we can conclude that

$$H_i^{(\alpha)} (s, \bar{s}, t) = \pm H_i^{(\alpha)} (\bar{s}, s, t). \quad (6)$$

Here and in the following the upper sign goes with the values $\alpha = 1, 2$, $i = 1, 2, 4$ and $\alpha = 3$, $i = 3$. The lower sign corresponds to $\alpha = 3$, $i = 1, 2, 4$ and $\alpha = 1, 2$, $i = 3$.

A. Reaction I. For photoproduction processes (reaction I) the isotopic functions in Eq. (2) are given by†

$$|i\rangle = N, \quad |f\rangle = \pi N. \quad (7)$$

Here N is the proton or neutron function:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (8)$$

and π is the pion function:

$$\pi^{\pm} = \mp \frac{1}{\sqrt{2}} (\delta_{\rho 1} \pm i\delta_{\rho 3}), \qquad \pi^0 = \delta_{\rho 3}. \quad (9)$$

The amplitudes

$$F^{(1/2)} = F^{(1)} + 2F^{(3)}, \qquad F^{(3/2)} = F^{(1)} - F^{(3)} \quad (10)$$

describe isotopic vector transitions into states with total isotopic spin ½ and $3/2$ respectively. $F^{(2)}$ describes the isotopic scalar transition into the state with isotopic spin ½.

In the barycentric frame of reaction I

$$s = (k + E_1)^2 = (\omega + E_2)^2, \quad \bar{s} = m^2 - 2k\,(E_2 + qx), \qquad t = \mu^2 - 2k\,(\omega - qx). \quad (11)$$

Here (and in the following) k and q are the magnitudes of the three dimensional photon and pion momenta, x is the cosine of the angle between these momenta, $\omega = \sqrt{\mu^2 + q^2}$ is the pion energy, and $E_1 = (m^2 + k^2)^{1/2}$ and $E_2 = (m^2 + q^2)^{1/2}$ are the energies of the initial and final nucleon.

The spin structure of F in the barycentric frame is

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*Our metric is such that $ab = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$, and our Feynman matrices are such that $\gamma^m \gamma^n + \gamma^n \gamma^m = 2g^{mn}$.

†With the usual normalization for the Clebsch-Gordan coefficients.

$$\mathcal{F} = \frac{m}{4\pi W} F = \chi^*{}_f \Big(i\sigma\mathbf{e}\mathcal{F}_1 + \frac{(\sigma\mathbf{q})(\sigma[\mathbf{ke}])}{qk}\mathcal{F}_2 + i\frac{(\sigma\mathbf{k})(\mathbf{qe})}{qk}\mathcal{F}_3 + i\frac{(\sigma\mathbf{q})(\mathbf{qe})}{q^2}\mathcal{F}_4\Big)\chi_i, \tag{12}*$$

where χ are two-component nucleon spinors. The $\mathcal{F}_i$ are functions of the total energy W and of x, and are related to the invariant functions H_i as follows:

$$\mathcal{F}_1 = C\frac{W+m}{2W}\Big[(W + E_2 + qx)\, H_2 + (\omega - qx)\, H_3 + \frac{4W}{W+m} H_4\Big],$$

$$\mathcal{F}_2 = C\frac{q(W-m)}{2W(E_2+m)}\Big[(W + E_2 + qx)\, H_2 + (\omega - qx)\, H_3 - \frac{4W}{W-m} H_4\Big], \tag{13}$$

$$\mathcal{F}_3 = -Cq\Big(\frac{W-m}{2} H_1 + H_2 - H_3\Big),$$

$$\mathcal{F}_4 = C\frac{q^2}{E_2+m}\Big(\frac{W+m}{2} H_1 - H_2 + H_3\Big);$$

$$C = [(W-m)/8\pi W]\, [(E_1+m)(E_2+m)]^{1/2}. \tag{14}$$

Their partial wave expansion is given by

$$\mathcal{F}_1 = \sum_{l=0}^{\infty} \{[lM_{l+} + E_{l+}]\, P'_{l+1}(x) + [(l+1)\, M_{l-} + E_{l-}]\, P'_{l-1}(x)\},$$

$$\mathcal{F}_2 = \sum_{l=1}^{\infty} [(l+1)\, M_{l+} + lM_{l-}]\, P'_l(x),$$

$$\mathcal{F}_3 = \sum_{l=1}^{\infty} \{[E_{l+} - M_{l+}]\, P''_{l+1}(x) + [M_{l-} + E_{l-}]\, P''_{l-1}(x)\},$$

$$\mathcal{F}_4 = \sum_{l=2}^{\infty} [M_{l+} - M_{l-} - E_{l+} - E_{l-}]\, P''_l(x). \tag{15}$$

Here $E_{l\pm}(M_{l\pm})$ correspond to pions produced with angular momentum l by an electric (magnetic) photon of angular momentum $l \pm 1$ in a state of total angular momentum $l \pm \frac{1}{2}$ and parity $-(-1)^l$.

It follows from the unitarity condition that below threshold for production of two pions we have

$$E_{l\pm} = |E_{l\pm}| \exp\{i\delta_{l\pm}\} \tag{16}$$

and analogously for $M_{l\pm}$, where $E_{l\pm}$, $M_{l\pm}$ are amplitudes for transitions with well defined isotopic spin (10), and $\delta_{l\pm}$ are the pion-nucleon scattering phase shifts in the corresponding states.

In what follows we shall need the connection between the partial amplitudes and the invariant functions H_i:

$$M_{l\pm} = C\int_{-1}^{1} dx \frac{1}{2} A_{l\pm}\Big[\frac{q(W-m)}{2}\frac{(1-x^2)P'_l(x)}{l(l+1)} H_1 + \Big\{\frac{W+m}{2W}(W + E_2 + qx)\Big[P_l(x) - \frac{q(W-m)P_{l\pm1}(x)}{(E_2+m)(W+m)}\Big] + q\frac{(1-x^2)P'_l(x)}{l(l+1)}\Big\}H_2 + \Big\{\frac{W+m}{2W}(\omega - qx) \times \Big[P_l(x) - \frac{q(W-m)P_{l\pm1}(x)}{(E_2+m)(W+m)}\Big] - q\frac{(1-x^2)P'_l(x)}{l(l+1)}\Big\}H_3 + 2\Big\{P_l(x) + \frac{qP_{l\pm1}(x)}{E_2+m}\Big\}H_4\Big],$$

$$E_{l\pm} = C\int_{-1}^{1} dx \frac{1}{2} B_{l\pm}\Big[\frac{q(1-x^2)}{2}\Big\{\frac{q(W+m)}{E_2+m} D_{l\pm} P'_{l\pm1}(x) - (W-m)\, A_{l\pm}P'_l(x)\Big\} H_1 + \Big\{\frac{W+m}{2W}(W + E_2 + qx)\Big[P_l(x) - \frac{q(W-m)P_{l\pm1}(x)}{(E_2+m)(W+m)}\Big] - q(1-x^2)\Big[A_{l\pm}P'_l(x) + D_{l\pm}\frac{qP'_{l\pm1}(x)}{E_2+m}\Big]\Big\}H_2 + \Big\{\frac{W+m}{2W}(\omega - qx)\Big[P_l(x) - \frac{q(W-m)P_{l\pm1}(x)}{(E_2+m)(W+m)}\Big] + q(1-x^2)\Big[A_{l\pm}P'_l(x) + D_{l\pm}\frac{qP'_{l\pm1}(x)}{E_2+m}\Big]\Big\}H_3 + 2\Big\{P_l(x) + \frac{qP_{l\pm1}(x)}{E_2+m}\Big\}H_4\Big], \tag{17}$$

where

$$A_{l\pm} = \begin{cases}(l+1)^{-1} \\ -l^{-1}\end{cases}, \quad B_{l\pm} = \begin{cases}(l+1)^{-1} \\ l^{-1}\end{cases}, \quad D_{l\pm} = \begin{cases}(l+2)^{-1} \\ (1-l)^{-1}\end{cases}.$$

The differential photoproduction cross section is given by

$$d\sigma/d\Omega = (q/k)\, |\mathcal{F}|^2. \tag{18}$$

B. Reaction III. For nucleon-antinucleon pair production processes (reaction III) the isotopic functions in Eq. (2) are given by

$$|i\rangle = \pi\bar{N}, \qquad |f\rangle = N, \tag{19}$$

where $\bar{N}$ is the antiproton or antineutron function:

$$\bar{p} = -\begin{pmatrix}1\\0\end{pmatrix}, \qquad \bar{n} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{20}$$

The amplitudes $F^{(1)}$ and $F^{(3)}$ correspond to isotopic vector transitions into states with total isotopic spin 0 and 1 respectively, $F^{(2)}$ corresponds to isotopic scalar transition into state with isotopic spin 1.

In the barycentric frame of reaction III we have

$$s = m^2 - 2\varkappa(E - py), \qquad \bar{s} = m^2 - 2\varkappa(E + py),$$
$$t = (\varkappa + \omega_\varkappa)^2 = 4E^2. \tag{21}$$

Here κ and p are the magnitudes of the photon and nucleon momenta, y = cos α is the cosine of the angle between these momenta, and E is the nucleon energy.

The spin structure of F in the barycentric frame is

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*[ke] = k × e; σk = σ · k.

$$F = \chi_N^* \left(i \frac{\mathrm{pe}}{p} \Phi_1 + \frac{[\mathrm{p}\,[\sigma \mathrm{p}]]\,[\varkappa \mathrm{e}]}{p^2 \varkappa} \Phi_2 + \frac{\sigma\,[\mathrm{pe}]}{p} \Phi_3 + \frac{\sigma\,[\varkappa \mathrm{e}]}{\varkappa} \Phi_4 \right) \chi_{\overline{N}}. \tag{22}$$

The Φ_i are functions of E and y and are related to the H_i by:

$$\Phi_1 = -\frac{2p\varkappa}{m} (E^2 H_1 + mH_2 + H_4),$$

$$\Phi_2 = \frac{2p^2\varkappa}{m} \left(H_2 + \frac{H_4}{E+m} \right),$$

$$\Phi_3 = (2p\varkappa E/m)\, H_3, \qquad \Phi_4 = -(2\varkappa E/m)\, H_4. \tag{23}$$

The amplitudes of Eq. (22) and the amplitudes $\Phi^{\nu}_{\lambda\overline{\lambda}}$ (the superscript specifies the helicity of the photon, the subscripts the helicities of the nucleon and antinucleon) for transitions between states with well defined helicities are related by:

$$\Phi^+_{\pm\pm} = \Phi^-_{\mp\mp} = \pm 2^{-1/2} (\Phi_1 \mp \Phi_4) \sin\alpha,$$

$$\Phi^+_{\pm\mp} = -\Phi^-_{\mp\pm} = 2^{-1/2} (\Phi_2 \pm \Phi_3 + \Phi_4)(1 \pm y). \tag{24}$$

According to Jacob and Wick the amplitudes (24) may be expanded as follows:

$$\Phi^+_{\pm\pm} = -\sum_J \sqrt{2}\,\Phi^J_{\pm\pm} \sin\alpha P'_J(y),$$

$$\Phi^+_{\pm\mp} = \sum_J \left[\frac{2}{J(J+1)} \right]^{1/2} \Phi^J_{\pm\mp} [(1 \mp y)\, P'_J(y) \pm J(J+1) P_J(y)]. \tag{25}$$

Therefore

$$\Phi_1 = \sum_J (\Phi^J_{--} - \Phi^J_{++})\, P'_J,$$

$$\Phi_2 = \sum_J \{ [J(J+1)]^{-1/2} [(\Phi^J_{-+} + \Phi^J_{+-})(P'_J + yP''_J) + (\Phi^J_{-+} - \Phi^J_{+-}) P''_J - (\Phi^J_{++} + \Phi^J_{--}) P'_J \},$$

$$\Phi_3 = -\sum_J [J(J+1)]^{-1/2} [(\Phi^J_{-+} + \Phi^J_{+-}) P''_J + (\Phi^J_{-+} - \Phi^J_{+-})(P'_J + yP''_J)],$$

$$\Phi_4 = \sum_J (\Phi^J_{--} + \Phi^J_{++})\, P'_J. \tag{26}$$

From Eqs. (6) and (23) it follows that for $\alpha = 1, 2$ the amplitudes $\Phi^J_{\pm\pm}$ and $\Phi^J_{-+} + \Phi^J_{+-}$ are different from zero only for odd values of J, and for $\alpha = 3$ only for even values; whereas the amplitude $\Phi^J_{-+} - \Phi^J_{+-}$ does not vanish for $\alpha = 1, 2$ for even J, and for $\alpha = 3$ for odd J.

The relation between the amplitudes $\Phi^J_{\lambda\overline{\lambda}}$ and the amplitudes for transitions between states with well defined parity is

$$\Phi^J_{--} - \Phi^J_{+-} = \mathscr{E}_{J-}, \quad \Phi^J_{--} + \Phi^J_{++} = J \mathscr{M}_{J-1,+} - (J+1) \mathscr{M}_{J+1,-},$$

$$\Phi^J_{-+} - \Phi^J_{+-} = [J(J+1)]^{1/2} \mathscr{E}_{J+},$$

$$\Phi^J_{-+} + \Phi^J_{+-} = [J(J+1)]^{1/2} (\mathscr{M}_{J-1,+} + \mathscr{M}_{J+1,-}), \tag{27}$$

where $\mathscr{E}_{l\pm}$ ($\mathscr{M}_{l\pm}$) describe the production by an electric (magnetic) photon of a nucleon-antinucleon pair with orbital angular momentum l, spin $\frac{1}{2} \pm \frac{1}{2}$ (1), and total angular momentum l ($l \pm 1$). The expansion (26) becomes therefore

$$\Phi_1 = \sum_l \mathscr{E}_{l-} P'_l, \quad \Phi_2 = \sum_l [\mathscr{M}_{l+} + \mathscr{M}_{l-} + \mathscr{E}_{l+}]\, P''_l,$$

$$\Phi_3 = -\sum_l [\mathscr{M}_{l+} P''_{l+1} + \mathscr{M}_{l-} P''_{l-1} + \mathscr{E}_{l+} (P'_l + yP''_l)],$$

$$\Phi_4 = \sum_l [(l+1) \mathscr{M}_{l+} P'_{l+1} - l \mathscr{M}_{l-} P'_{l-1}]. \tag{28}$$

For $\alpha = 1, 2$ the amplitudes $\mathscr{M}_{l\pm}$, $\mathscr{E}_{l+}$ have l even and $\mathscr{E}_{l-}$ has l odd; whereas for $\alpha = 3$ $\mathscr{E}_{l-}$ has l even and $\mathscr{M}_{l\pm}$, $\mathscr{E}_{l+}$ have l odd.

Let us consider the unitarity condition for reaction III with only the two-pion intermediate state taken into account. For the isotopic scalar amplitudes with odd total angular momentum J we find

$$\mathrm{Im}\, \Phi^{(2)J}_{\lambda\overline{\lambda}} = (q_1 \varkappa / 16\pi)\, T^{(-)\,J}_{\lambda\overline{\lambda}} f_J^*, \tag{29}$$

where $q_1 = (E^2 - \mu^2)^{1/2}$ is the pion momentum in the intermediate state,

$$T^{(-)J}_{\pm\pm} = T^{(-)J}_{+}, \qquad T^{(-)J}_{\pm\mp} = T^{(-)J}_{-} \tag{30}$$

represent the amplitudes for the process $\pi\pi \rightarrow N\overline{N}$, determined by Eqs. (3.13) — (3.17) of Frazer and Fulco,[15] and f_J is the amplitude for the process $\gamma\pi \rightarrow \pi\pi$, determined by Eq. (6) of Solov'ev, who gives an explicit formula for f_1.[13]

The two-pion contribution vanishes for the isotopic vector amplitudes $\Phi^{(1,3)}$. It follows from Eqs. (27), (29), and (30) that the two-pion intermediate state contributes only to the isotopic scalar magnetic transitions of reaction III:

$$\mathrm{Im}\, (\Phi^{(2)J}_{\substack{--\\-+}} + \Phi^{(2)J}_{\substack{++\\+-}}) = \frac{q_1 \varkappa}{8\pi} T^{(-)J}_{\pm} f_J^*. \tag{31}$$

3. SPECTRAL REPRESENTATIONS AND PHOTOPRODUCTION EQUATIONS

As remarked in Sec. 1 we shall assume that spectral representations without subtractions are valid for the invariant photoproduction amplitudes. The Mandelstam representation[11] takes the form:

$$H_i^{(\alpha)}(s, \bar{s}, t) = \left(B_i^{(\alpha)} - \frac{2eg\delta_{i1}}{t - \mu^2} \right) \left(\frac{1}{s - m^2} \pm \frac{1}{\bar{s} - m^2} \right) + \frac{1}{\pi^2} \int_{(m+\mu)^2}^{\infty} dx \int_{4\mu^2}^{\infty} dy \left(\frac{1}{x - s} \pm \frac{1}{x - \bar{s}} \right) \frac{h_{i1}^{(\alpha)}(x, y)}{y - t} + \frac{1}{\pi^2} \iint_{(m+\mu)^2}^{\infty} dx\, dy \frac{h_{i2}^{(\alpha)}(x, y)}{(x - s)(y - \bar{s})}, \qquad (32$$

$$h_{i2}^{(\alpha)}(x, y) = \pm h_{i2}^{(\alpha)}(y, x). \tag{33}$$

The signs in Eqs. (32) and (33) correspond to the signs in Eq. (6):

$$B_1^{(\alpha)} = 0, \quad B_2^{(\alpha)} = B_3^{(\alpha)} = \frac{g}{2}\begin{cases}\mu_p' - \mu_n, & \alpha = 1\\ \mu_p' + \mu_n, & \alpha = 2,\\ \mu_p' - \mu_n, & \alpha = 3\end{cases}$$

$$B_4^{(\alpha)} = -\tfrac{1}{4}eg - mB_2^{(\alpha)} \tag{34}$$

(rational units). In contrast to pion-nucleon scattering in photoproduction one obtains a contribution from the single pion state, related by gauge invariance to the single nucleon contribution.

We restrict ourselves to low energies and consider the approximate representation of Cini and Fubini.[2,3] If the amplitudes F, H, etc., for photoproduction of pions on pions are ignored then the representation takes the form

$$H_i^{(\alpha)}(s, \bar{s}, t) = \text{Born term}$$

$$+ \frac{1}{\pi}\int\limits_{(m+\mu)^2}^{\infty} dx\left(\frac{1}{x-s} \pm \frac{1}{x-\bar{s}}\right) a_i^{(\alpha)}(x, t)$$

$$+ \delta_{\alpha 2}\frac{1 \pm 1}{2\pi}\int\limits_{4\mu^2}^{\infty}\frac{b_i(x)\,dx}{x-t}, \tag{35}$$

where

$$a_i^{(\alpha)} = \operatorname{Im} H_i^{(\alpha)} \text{ for reaction I}, \quad \delta_{\alpha 2} b_i = \operatorname{Im} H_i^{(\alpha)} \text{ for reaction III}. \tag{36}$$

The $a_i^{(\alpha)}(s,t)$ correspond to only the pion-nucleon intermediate state in the unitarity condition. Consequently their singularities in t do not start until $16\mu^2$ and they may therefore be expanded in a Taylor series. If the D, F, etc., pion-nucleon scattering phase shifts are ignored then this expansion takes the form

$$a_i^{(\alpha)}(s, t) = a_{i0}^{(\alpha)}(s) + (t - t_0)\, a_{i1}^{(\alpha)}(s). \tag{37}$$

To make sure that the functions $a_i^{(\alpha)}(s,t)$ do not contain unobservable angles it is necessary to take for t_0 the threshold value of t:

$$t_0 = \mu^2 - 2\mu k_{\text{th}}, \qquad k_{\text{th}} = \mu\,(2m + \mu)\,/2\,(m + \mu). \tag{38}$$

With the help of Eqs. (36), (11), (13) and (15) the functions $a_{i0}^{(\alpha)}$ and $a_{i1}^{(\alpha)}$ are easily expressed in terms of the imaginary parts of the partial photoproduction amplitudes. If we then substitute Eq. (35) into Eq. (17) we obtain the integral equation for photoproduction.

A. Isotopic vector amplitudes. The last integral in Eq. (35) is absent for the isotopic vector amplitudes ($\alpha = 1, 3$), and we obtain relations previously discussed[9–11] by expanding in powers of ω/m and ω'/m and retaining only the leading terms. The nucleon recoil must be taken into account more accurately for the resonant magnetic dipole amplitude, and this is easy to do with the help of Eqs. (17) and (35).

B. Isotopic scalar amplitudes. For the isotopic scalar amplitudes ($\alpha = 2$) reaction III contributes to Eq. (35). From Eqs. (36), (23), (26) and (31) we obtain the following expressions for $b_i(t)$:

$$b_1(t) = \frac{mq_1}{16\pi E p^2}\left[T_+^{(-)1} - \frac{m}{\sqrt{2}E}\,T_-^{(-)1}\right] f_1^*,$$

$$b_2(t) = \frac{mq_1}{16\pi p^2}\left[-\frac{m}{E}\,T_+^{(-)1} + \frac{1}{\sqrt{2}}\,T_-^{(-)1}\right] f_1^*,$$

$$b_3(t) = 0, \quad b_4(t) = -(mq_1/16\pi E)\,T_+^{(-)1} f_1^*, \tag{39}$$

where, as noted above, the amplitudes $T_\pm^{(-)1}$ have been discussed by Frazer and Fulco[15] and f_1 by Solov'ev.[13]

For the amplitude for photoproduction of pions we have[13]

$$f_1(t) = \Lambda e^{i\delta(t)}\varphi(t), \tag{40}$$

where $\delta \equiv \delta_1^1$ is the pion-pion scattering phase shift in the $J = I = 1$ state, and $\varphi(t)$ is a known integral containing this phase shift [Eqs. (36), (38) of Solov'ev[13]]. The parameter Λ is given by[13]

$$\Lambda = \frac{2}{\pi}\int\limits_{4m^2}^{\infty}\frac{ds'}{s'}\left(\operatorname{Im}\mathcal{F}(s', \cos\theta = 1)\right)_{N\bar{N}}, \tag{41}$$

where the function appearing under the integral sign is related to the amplitudes for the processes $\gamma\pi \to N\bar{N}$ and $\pi\pi \to N\bar{N}$ as follows:

$$(\operatorname{Im}\mathcal{F}(s', \cos\theta))_{N\bar{N}} = \sum_{J\ \text{odd}} (\operatorname{Im} f_J(s'))_{N\bar{N}} P_J'(\cos\theta), \tag{42}$$

$$(\operatorname{Im} f_J)_{N\bar{N}} = \frac{m^2}{\pi s'^2}\frac{p'}{q_1'}$$

$$\times\left[T_+^{(-)J*}\left(\Phi_{++}^{(2)J} + \Phi_{--}^{(2)J}\right) + T_-^{(-)J*}\left(\Phi_{-+}^{(2)J} + \Phi_{+-}^{(2)J}\right)\right];$$

$$s'^2 = 2\,(p'^2 + m^2) = 2(q_1'^2 + \mu^2). \tag{43}$$

The amplitudes that enter here correspond to high energies and cannot be calculated at present. Therefore the parameter Λ must be taken from experiment.

If it is assumed that the phase shift δ_1^1 has a resonance then the function φ in Eq. (40) also has a resonant character.[13] If one also ignores the nonresonant pion-nucleon scattering phase shifts, then one obtains from Eq. (35) the following expressions for the isotopic scalar photoproduction amplitudes:

$$H_i^{(2)}(s, \bar{s}, t) = \left(B_i^{(2)} - \frac{2eg\delta_{i1}}{t - \mu^2}\right)\left(\frac{1}{s - m^2} \pm \frac{1}{\bar{s} - m^2}\right)$$

$$+ \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\frac{b_i(t')\,dt'}{t' - t}, \tag{44}$$

where the b_i are given by Eq. (39). The π^+ to π^- photoproduction cross sections ratio at threshold should depend sensitively on the magnitude of expression (44).

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A MODEL OF LOCAL FIELD THEORY WITH FINITE CHARGE RENORMALIZATION

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The local field theory proposed by Bialynicki-Birula is treated by a previously developed method.[3,4] The S matrix and renormalization constants are derived. It is proved that the charge renormalization is finite in all orders and does not contain logarithmic singularities. It is shown that the ultraviolet contribution results in a series with a finite sum, and that the Green's function series converges absolutely at low values of the time and has a branch point at $t = 0$, the singularity at zero being integrable.

INTRODUCTION

In view of the still unsurmounted difficulties in the solution of the exact field equations in quantum field theory, the use of various models for an investigation of the intrinsic consistency of this theory has become quite popular. An investigation of the Lee model[1] has seemingly led to the conclusion that the theory is intrinsically inconsistent. It was shown later, however,[2] that the model was not self consistent because of the simplifications made to obtain an exactly solvable Hamiltonian. In particular, these simplifications violated the important requirement of crossing symmetry.

In the present article we investigate, by a method previously developed,[3,4] the modified Lee model proposed by Bialynicki-Birula,[5] in which the crossing symmetry condition is satisfied. The solutions are obtained in the form of series in the renormalized constant Δm (Δm is the physical parameter causing the mass difference between two fermion states in the model). The convergence of this series is proved for the ultraviolet region $E \gg \Delta m$. An important property of the model is the finite renormalization, in the case of a point interaction, of the charge in all orders in Δm, in contrast with the Lee model, where the well known zero-charge problem exists.

The Green's function of the model considered has all the properties of the Green's function of the renormalized theory.[6,7] Specifically, the renormalized Green's function is analytic in the t plane and has a branch point at $t = 0$; when $g^2/\pi^2 < 1$ there exists a Fourier transform of the Green's function, which admits of an expansion about the point $g^2 = 0$.

1. THE S MATRIX OF THE MODEL

Bialynicki-Birula[5] has considered a model of local field theory with fixed nucleon, in which the nucleon can be in two states of different mass (we shall call these, arbitrarily, proton and neutron states). The Hamiltonian of the system has the form

$$H = m_0(\psi^+\psi) + \frac{1}{2}\int dx : [\pi^2(x) + (\nabla\varphi(x))^2 + \mu^2\varphi^2(x)]:$$

$$+ g(\psi^+\tau_1\psi)\int dx\, \varphi(x)\,\delta(x) + \Delta m_0(\psi^+\tau_3\psi), \qquad (1)$$

where $\psi = v_p c_p + v_n c_n$ is the nucleon field operator, c_N ($N = p, n$) is the nucleon annihilation operator, v_N is a spinor describing the nucleon,

$$v_p = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v_n = \begin{pmatrix}0\\1\end{pmatrix},$$

$\pi(x)$ and $\varphi(x)$ are the meson field operators, and τ_1 and τ_3 are the matrices of isotopic spin $1/2$.

Noting that when $\Delta m_0 = 0$ we obtain the exactly solvable case of scalar mesons with fixed source, it is possible to consider perturbations in the constant Δm_0 without limiting the interaction forces between the nucleon and the mesons. By this method, Bialynicki-Birula[5] obtained the interesting result that the renormalization of the charge is finite and contains no logarithmic singularities. From the point of view of the method which we have developed earlier,[3,4*] the Hamiltonian (1) is of interest because the Lappo-Danilevskii series coincides here with the perturbation-theory series in the constant Δm_0, but the new method, unlike perturbation theory, makes it possible to obtain the n-th term of the series.

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*We denote throughout reference 4 by I, and the corresponding formulas in this paper will be designated (I, 4.2) etc.

Thus, let us consider the equation for the S matrix in the interaction representation. We seek directly the "adiabatic" S^{α} matrix, so as to use formulas (I, 4.2) and (I, 4.3). In the interaction representation we have

$$i\partial S^{\alpha}(t,t_0)/\partial t = H_I(t)e^{-\alpha|t|}S^{\alpha}(t,t_0), \quad S^{\alpha}(t,t_0)|_{t=t_0}=1,$$

$$H_I(t)=g(\psi^+\tau_1\psi)\hat{\varphi}(t)+\Delta m_0(\psi^+\tau_3\psi),$$

$$\hat{\varphi}(t)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega}}(a_{\mathbf{k}}e^{-i\omega t}+a_{\mathbf{k}}^{+}e^{i\omega t}). \tag{2}$$

Repeating in its entirety the procedure developed in I, (Secs. 1 — 3), we obtain the following expression for the $S^{\alpha}(t, t_0)$ matrix:

$$S^{\alpha}(t,t_0)=1-[2(\psi^+\psi)-(\psi^+\psi)^2]$$
$$+\sum_{q=0}^{\infty}\frac{[-i(\psi^+\tau_3\psi)\Delta m_0]^q}{q!}\int_{t_0}^{t}d\xi_1\ldots\int_{t_0}^{t}d\xi_q$$
$$\times:\exp\left\{-i(\psi^+\tau_1\psi)g\int_{t_0}^{t}ds\prod_{j=1}^{q}\varepsilon(\xi_j-s)\hat{\varphi}(s)e^{-\alpha|s|}\right\}:$$
$$\times\exp\left\{-\frac{ig^2}{2}\int_{t_0}^{t}\int_{t_0}^{t}ds_1\,ds_2e^{-\alpha(|s_1|+|s_2|)}\right.$$
$$\left.\times\prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\Delta(s_1-s_2)\varepsilon(\xi_j-s_2)\right\}$$
$$\times\exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j|\right\}. \tag{3}$$

The resultant S^{α} matrix is defined accurate to a phase-shift factor, and satisfies the equations

$$S^{\alpha}(\infty,-\infty)|0\rangle=|0\rangle,$$
$$S^{\alpha}(\infty,-\infty)|N\rangle=e^{i\,\mathrm{const}/\alpha}|N\rangle, \tag{4}$$

where $|0\rangle$ and $|N\rangle$ denote the vacuum and single-nucleon states of the zero-order Hamiltonian. The usual conditions of stability of the vacuum and of the single-particle states require,[8] however, that

$$S(\infty,-\infty)|0\rangle=|0\rangle, \qquad S(\infty,-\infty)|N\rangle=|N\rangle. \tag{5}$$

In all further calculations we shall use the adiabatic hypothesis, which enables us, in particular, to eliminate correctly the phase-shift factor, which is indeterminate when $\alpha = 0$ (reference 9), and consequently to satisfy condition (5) (see Appendix A).

We consider now the most important matrix elements of the S matrix. The eigenvalue of the energy of the single-fermion state is[9] (see Appendix B)

$$E_N=\lim_{\alpha\to 0}\frac{\langle N|HS^{\alpha}(0,-\infty)|N\rangle}{\langle N|S^{\alpha}(0,-\infty)|N\rangle}$$
$$=m+\delta_N\Delta m\sum_{q=0}^{\infty}(-\delta_N\Delta m)^q\int_0^{\infty}dx_1\ldots\int_0^{\infty}dx_qx_1\ldots x_q$$
$$\times\frac{\partial^q}{\partial x_1\ldots\partial x_q}\exp\left\{2g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\sum_{l=1}^{q}\sum_{m=1}^{l}\right.$$
$$\left.\times(-1)^{l+m}\exp\left(-\omega\sum_{j=m}^{l}x_j\right)\right\},$$

$$\delta_N=\begin{cases}+1 \text{ for the proton } (N=p)\\ -1 \text{ for the neutron } (N=n).\end{cases} \tag{6}$$

Further

$$m=m_0-\frac{1}{2}g^2\sum_{\mathbf{k}}\omega^{-2}, \tag{6$'$}$$

$$\Delta m=\Delta m_0\exp\left\{-g^2\sum_{\mathbf{k}}\omega^{-3}\right\}. \tag{6$''$}$$

The requirement that the observed m and Δm be finite causes us to consider the bare quantities m_0 and Δm_0 as infinite. We note that the renormalization of m_0 coincides precisely with the renormalization for the case of scalar mesons in a field of a fixed source. After carrying out the renormalizations (6′) and (6″), each term of the series (6) is finite subject to the condition* $g^2/\pi^2 < 1$ (see Appendix B for details).

The renormalization constant of the fermion field Z_2^N is determined, in accordance with its probabilistic sense, by the square of the matrix elements

$$Z_2^N=|\langle N|\mathbf{N}\rangle|^2=|\langle N|S^{\alpha}(0,-\infty)|N\rangle|^2$$
$$=Z_2^{\mathrm{sc}}\left[\sum_{q=0}^{\infty}(\delta_N\Delta m)^q\int_0^{\infty}dx_1\ldots\int_0^{\infty}dx_qx_1\ldots x_q\right.$$
$$\times\frac{\partial^q}{\partial x_1\ldots\partial x_q}\exp\left\{g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\left[-\sum_{l=1}^{q}(-1)^l\exp\left(-\omega\sum_{j=1}^{l}x_j\right)\right.\right.$$
$$\left.\left.\left.+2\sum_{l=2}^{q}\sum_{m=2}^{l}(-1)^{l+m}\exp\left(-\omega\sum_{j=m}^{l}x_j\right)\right]\right\}\right]^2. \tag{7}$$

Here

$$Z_2^{\mathrm{sc}}=\exp\left\{-\frac{1}{2}g^2\sum_{\mathbf{k}}\omega^{-3}\right\}$$

is the value of Z_2 for scalar mesons, and $|\mathbf{N}\rangle$ denotes the single-nucleon state of the complete Hamiltonian.

Let us emphasize that the constant Z_2^N in this model is equal to the product of the constant Z_2^{sc} of the scalar neutral theory with fixed source and a series of finite terms, so that when $g^2/\pi^2 < 1$ all the terms of the series are finite. Later on this circumstance will enable us to draw some conclusions regarding the constant Z_1.

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*Bialynicki-Birula[5] gives an incorrect condition for the finiteness of the terms of the series $g^2/4\pi^2 < e^{-1}$.

The renormalized coupling constant is defined as usual:

$$\frac{g_r}{g} = \langle p \,|\, \psi^+ \tau_1 \psi \,|\, n$$

$$= \lim_{\alpha \to 0} \frac{\langle p \,|\, S^\alpha(\infty, 0)\, \psi^+ \tau_1 \psi S^\alpha(0, -\infty) \,|\, n \rangle}{[\langle p \,|\, S^\alpha(\infty, -\infty) \,|\, p \rangle \langle n \,|\, S^\alpha(\infty, -\infty) \,|\, n \rangle]^{1/2}}$$

$$= 1 - \sum_{q=1}^{\infty} (\Delta m)^{2q} \int_0^\infty dx_1 \ldots \int_0^\infty dx_{2q-1}\, x_1 \ldots x_{2q-1} \sum_{j=1}^{q} x_{2j-1}$$

$$\cdot \frac{\partial^{2q-1}}{\partial x_1 \ldots \partial x_{2q-1}}$$

$$\times \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{2q-1} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-\omega \sum_{j=m}^{l} x_j\right)\right\}. \quad (8)$$

The situation in this model differs appreciably from that obtaining in the charge theory [see (I, 4.10) ff.], for at the point $g^2 = 0$ all the integrals are bounded, in contrast with expression (I, 4.12). Therefore, when perturbation theory is used here, i.e., when the solution is represented as a series in g^2, no logarithmic divergences in the maximum cut-off momentum takes place, such as occur in the local field theory. In this connection the given model does not reflect, in our opinion, some of the fundamental difficulties inherent in the exact equations of mesodynamics.

The renormalization constant of the vertex part Z_1 can be obtained from the well known equation

$$g_r = Z_1^{-1} (Z_2^p Z_2^n)^{1/2} Z_3^{1/2} g.$$

In our case the renormalization constant of the meson field Z_3 is equal to unity. Using (7) and (8), we can conclude that the constant Z_1 has a structure

$$Z_1 = Z_2^{\mathrm{sc}} \sigma(g^2, \Delta m), \quad (9)$$

where $\sigma(g^2, \Delta m)$ is a series in Δm, all the terms of which are finite when $g^2/\pi^2 < 1$.

Let us write down the matrix element for the elastic scattering of a meson by a nucleon (see Appendix A):

$$S_{f \leftarrow i} = \lim_{\alpha \to 0} \frac{\langle N \,|\, a_{\mathbf{p}_f} S^\alpha(\infty, -\infty)\, a^+_{\mathbf{p}_i} \,|\, N \rangle}{\langle N \,|\, S^\alpha(\infty, -\infty) \,|\, N \rangle} = \delta(\mathbf{p}_i - \mathbf{p}_f) - 2\pi i \delta(\omega_i - \omega_f)\, M_{f \leftarrow i}(\omega_f);$$

$$M_{f \leftarrow i}(\omega_f) = -\frac{2\delta_N g^2}{\omega_f^2} \frac{\Delta m}{\omega_f} \sum_{q=0}^{\infty} (-i\delta_N \Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \left[q + 1 - \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \left\{\exp\left(i\omega_f \sum_{j=m}^{l} x_j\right)\right.\right.$$

$$\left.\left. + \exp\left(-i\omega_f \sum_{j=m}^{l} x_j\right)\right\}\right] (-1)^q \int_{x_1}^\infty dy_1 \ldots \int_{x_q}^\infty dy_q \frac{\partial^q}{\partial y_1 \ldots \partial y_q} \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-i\omega \sum_{j=m}^{l} y_j\right)\right\}. \quad (10)$$

2. ON THE CONVERGENCE OF THE SERIES FOR E_p, Z_2^n, AND g_r

The proof of the convergence of the series (6) – (10) is exceedingly complicated, since an estimate of the n-th term calls for extremely fine approximation methods, without which one cannot judge the behavior of the series as a whole. However, the series for E_p, Z_2^n, and g_r can be summed in the generalized sense[10] (we shall refer to this as the ϵ-sense), the gist of which we shall explain by using the series for E_p as an example.

Let us consider the series

$$\delta m = \sum_{q=0}^{\infty} (-\Delta m)^q \int_0^\infty dx_1 \ldots$$

$$\times \int_0^\infty dx_q\, x_1 \ldots x_q \frac{\partial^q}{\partial x_1 \ldots \partial x_q} F_q(x_1, \ldots, x_q),$$

$$F_q(x_1, \ldots, x_q)$$

$$= \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-\omega \sum_{j=m}^{l} x_j\right)\right\}. \quad (11)$$

After integrating by parts in each term, we obtain (see formula (B, 8) in Appendix B)

$$\delta m = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \prod_{j=1}^{q} (1 - \hat{Q}_j)\, F_q(x_1, \ldots, x_q), \quad (12)$$

where the operator $\hat{Q}_j$ is defined by the equation

$$\hat{Q}_j F_q(\ldots, x_j, \ldots,) = F_q(\ldots, \infty, \ldots).$$

We consider a different series

$$\delta m^\varepsilon = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \exp\left(-\varepsilon \sum_{j=1}^{q} x_j\right)$$

$$\times \prod_{i=1}^{q} (1 - \hat{Q}_i)\, F_q(x_1, \ldots, x_q), \quad (13)$$

which becomes equal to (12) when $\epsilon = 0$. It can be shown that when ϵ satisfies the inequality

$$\frac{\Delta m}{\varepsilon} < 1 - \Delta m \int_0^\infty dx e^{-\varepsilon x} F_1(x), \quad (14)$$

the series (13) is bounded by the quantity (see Appendix B)

$$\delta m^{\varepsilon} < \left\{1 - \Delta m \int_0^{\infty} dx e^{-\varepsilon x} [F_1(x) - 1]\right\}^{-1}. \tag{15}$$

As $\epsilon \rightarrow 0$, the left half of (15) goes into the original series (12), while the right half has a finite positive limit when

$$\Delta m \int_0^{\infty} dx\, [F_1(x) - 1] < 1.$$

This procedure signifies that the series (11) is summable in the general sense* (ϵ-sense) when

$$\Delta m \int_0^{\infty} dx\, [F_1(x) - 1] < 1$$

and its sum (in the ϵ-sense) is bounded by the quantity

$$\delta m < \left\{1 - \Delta m \int_0^{\infty} dx\, [F_1(x) - 1]\right\}^{-1}. \tag{16}$$

The series for Z_2^n and g_r are also summed in the ϵ-sense. We cite the result for the sums (in the ϵ-sense)

$$Z_2^n < Z_2^{sc} \left\{1 - \Delta m \int_0^{\infty} dx\, [F_1(x) - 1]\right\}^{-2}, \tag{17}$$

$$\left|\frac{g_r}{g} - 1\right| < 2(\Delta m)^2 \int_0^{\infty} dx\, x [F_1(x) - 1] \left\{1 - \left(\Delta m \int_0^{\infty} dx\, [F_1(x) - 1]\right)^2\right\}^{-2}. \tag{17$'$}$$

Let us discuss the physical meaning of summation in the ϵ-sense. We have seen that the series (13) converges in the usual manner at sufficiently large ϵ [condition (14)] but the series (13) differs from (12) in that the contribution from the large values of the time (large x_j), i.e., from low energies, is smaller. Consequently, the contribution from the ultraviolet region is found to be summable to a finite quantity, which is unexpected from the point of view of local field theory. Thus, the convergence of the series in the ϵ-sense presupposes that the contribution from large times (low energies) is not large enough to violate the summability of expressions (6) — (10).

In this connection, it is interesting to investigate the behavior of the Green's function for small terms, i.e., in the ultraviolet region. The Green's function is defined as

$$G(t) = \langle 0 | T\{\psi(t)\psi^+(0) S(\infty, -\infty)\} | 0 \rangle = Z_2^{sc} G_{sc}(t) \sum_{q=0}^{\infty} (-i\tau_3 \Delta m)^q \int_0^t d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{q-1}} d\xi_q \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3}\right.$$

$$\left. \times \left[\sum_{l=1}^{q} (-1)^l (e^{-i\omega(t-\xi_l)} - e^{-i\omega\xi_l}) + 2 \sum_{l=2}^{q} \sum_{m=1}^{l-1} (-1)^{l+m} e^{-i\omega(\xi_m - \xi_l)}\right]\right\},$$

$$G_{sc}(t) = \theta(t) e^{-imt} \exp\left\{\frac{1}{2} g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} e^{-i\omega t}\right\}. \tag{18}$$

The function $G_{sc}(t)$ is the Green's function of the scalar neutral theory, and behaves as $t^{-g^2/4\pi^2}$ at small values of the time.[11]

It can be shown (see Appendix D) that the series in Δm converges absolutely at small values of t ($\mu t \ll 1$), provided that $g^2/\pi^2 < 1$ and has a branch point at $t = 0$. Thus, the function $G(t)$ has for $t = 0$ a singularity which is integrable because of the condition $g^2/\pi^2 < 1$, which makes the Fourier transform possible.

CONCLUSION

The proposed example of a model with finite charge renormalization shows another possibility (compared with the Lee model) which can be realized in the rigorous theory. In our opinion, however, this model also fails to reflect the true situation in field theory, since an investigation of more complicated models (see reference 4) leads apparently to a conclusion that the exact solution has a singularity $g^2 = 0$, a Taylor-series expansion about which leads to additional divergences. As already noted above, the model considered here does not have such a property.

In conclusion, the authors express deep gratitude to Prof. D. I. Blokhintsev for continuous interest in the work and for stimulating discussions, and also to L. G. Zastavenko for a discussion of the mathematical problems.

APPENDIX A

Since the S^{α} matrix is specified in the form of a series, the matrix elements are presented in the form of a limit of a ratio of two series as $\alpha \rightarrow 0$. It is found that if one series is divided by the other and terms of equal powers of Δm collected, the phase shift cancels out in the terms obtained in this manner and consequently one can go in each term separately to the limit as $\alpha \rightarrow 0$.

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*The method proposed is regular, i.e., convergent series such as (11) are summed to their usual sum.[10]

Let us illustrate this method of eliminating the phase shift by an example of the matrix element of the scattering of a meson by a nucleon [see formula (10)]:

$$M_{f\leftarrow i}(\omega_f) = \frac{g^2}{2i\omega_f}\int_{-\infty}^{\infty} d\tau e^{-i\omega_f \tau} \lim_{\alpha\to 0} \frac{\sum_{q=1}^{\infty}(-i\delta_N \Delta m_0)^q B_q^\alpha(\tau)}{\sum_{q=0}^{\infty}(-i\delta_N \Delta m_0)^q b_q^\alpha}, \tag{A.1}$$

$$B_q^\alpha(\tau) = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_q$$
$$\times \exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j|\right\}\prod_{l=1}^{q}\varepsilon(\xi_l)\,\varepsilon(\xi_l-\tau)$$
$$\times\ \exp\left\{-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2 e^{-\alpha(|s_1|+|s_2|)}\right.$$
$$\left.\times\prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)\right\}, \tag{A.2$'$}$$

$$b_q^\alpha = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \dots \int_{-\infty}^{\infty} d\xi_q \exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j|\right.$$
$$-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2 e^{-\alpha(|s_1|+|s_2|)}$$
$$\left.\times\prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)\right\}. \tag{A.2$''$}$$

The integral in the exponential function is

$$-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2 \exp\{-\alpha(|s_1|+|s_1|)\}$$
$$\times\prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)$$
$$= -\frac{g^2}{2i\alpha}\sum_{\mathbf{k}}\frac{1}{\omega^2} - g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\left\{q + 2\sum_{\nu_1=2}^{q}\sum_{\nu_2=1}^{\nu_1-1}\prod_{j_1\neq\nu_1}\varepsilon(\xi_{\nu_1}-\xi_{j_1})\right.$$
$$\left.\times\prod_{j_2\neq\nu_2}\varepsilon(\xi_{\nu_2}-\xi_{j_2})\exp(-i\omega|\xi_{\nu_1}-\xi_{\nu_2}|)\right\}. \tag{A.3}$$

The first term is the same for all terms of the series and therefore cancels out.

The presence of a phase shift in the expression for the matrix element manifests itself in the fact that the integrals with respect to time (with respect to ξ_j) diverge linearly at infinity when $\alpha = 0$. However, as stated above, a series that represents a ratio of series that diverge when $\alpha = 0$ contains integrals that are finite when $\alpha = 0$. This circumstance is due to the fact that the divergent integrals are grouped about each power of Δm in such a way, that the infinities cancel out.

The regularization procedure is as follows. We change in the integrals (A.2′) and (A.2″) to integration over the simplex and make a change of variables

$$\xi_1 = \xi,\quad \xi_l = \xi - \sum_{j=1}^{l-1} x_j,\qquad l \geqslant 2. \tag{A.4}$$

Then

$$B_q^\alpha(\tau) = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \dots \int_0^{\infty} dx_{q-1}$$
$$\times\exp\left\{-\alpha\left(|\xi| + \sum_{l=1}^{q-1}\left|\xi - \sum_{j=1}^{l} x_j\right|\right)\right\}$$
$$\times\ \varepsilon(\xi)\,\varepsilon(\xi-\tau)\prod_{l=1}^{q-1}\varepsilon\left(\xi - \sum_{j=1}^{l} x_j\right)\varepsilon\left(\xi-\tau-\sum_{j=1}^{l} x_j\right)$$
$$\times\ F_{q-1}(x_1, \dots, x_{q-1})\,(\Delta m/\Delta m_0)^q$$

$$b_q^\alpha = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \dots \int_0^{\infty} dx_{q-1}$$
$$\times\exp\left\{-\alpha\left(|\xi| + \sum_{l=1}^{q-1}\left|\xi - \sum_{j=1}^{l} x_j\right|\right)\right\}$$
$$\times\ F_{q-1}(x_1, \dots, x_{q-1})\,(\Delta m/\Delta m_0)^q \tag{A.5}$$

where

$$F_{q-1}(x_1, \dots, x_{q-1})$$
$$= \exp\left\{2g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\sum_{l=1}^{q-1}\sum_{m=1}^{l}(-1)^{l+m}\exp\left(-i\omega\sum_{j=m}^{l} x_j\right)\right\}.$$

The integral of the function F_{q-1} diverges linearly at infinity with respect to any argument x_j, but the function $F_{q-1}(\dots, x_j, \dots) - F_{q-1}(\dots, \infty, \dots)$ is already integrable with respect to x_j at infinity. We note further that

$$F_{q-1}(x_1, \dots, x_{j-1}, \infty, x_{j+1}, \dots, x_{q-1})$$
$$= F_{j-1}(x_1, \dots, x_{j-1})\,F_{q-j-1}(x_{j+1}, \dots, x_{q-1}). \tag{A.6}$$

It follows from the foregoing that in order to regularize the function $B_q^\alpha(\tau)$ it is necessary to subtract from the function F_{q-1} its value at infinity in each of its arguments, i.e., we substitute instead of F_{q-1} the function

$$\prod_{l=1}^{q-1}(1-\hat{Q}_j)\,F_{q-1}(x_1, \dots, x_{q-1}), \tag{A.7}$$

where the operator $\hat{Q}_j$ is defined by the equation $\hat{Q}_j F_{q-1}(\dots, x_j, \dots) = F_{q-1}(\dots, \infty, \dots)$. Such a substitution of (A.7) for the function F_{q-1} is realized precisely by dividing one series by the other and grouping the terms about equal powers of Δm. The same applies also to the regularization of the integral with respect to ξ, (A.5). Finally, the scattering matrix element with the phase shift eliminated is written in the form

$$M_{f\leftarrow i}(\omega_f)=\frac{g^2}{2i\omega_f}\int\limits_{-\infty}^{\infty}d\tau e^{-i\omega_f\tau}\sum_{q=1}^{\infty}(-i\delta_N\Delta m)^q\int\limits_0^{\infty}dx_1\ldots\int\limits_0^{\infty}dx_q$$
$$\times\prod_{j=1}^{q}(1-\hat{Q}_j)F_q(x_1,\ldots,x_q)\int\limits_{-\infty}^{\infty}d\xi\,[\varepsilon(\xi)\varepsilon(\xi-\tau)$$
$$\times\prod_{l=1}^{q}\varepsilon\Bigl(\xi-\sum_{j=1}^{l}x_j\Bigr)\varepsilon\Bigl(\xi-\tau-\sum_{j=1}^{l}x_j\Bigr)-1\Bigr]. \qquad (A.8)$$

From (A.8) we obtain (10) directly if we recognize that the operator $(1-\hat{Q}_j)$ can be represented in the form

$$1-\hat{Q}_j=-\int\limits_{x_j}^{\infty}dy_j\frac{\partial}{\partial y_j}. \qquad (A.9)$$

The phase shift for any other matrix element is eliminated analogously.

APPENDIX B

According to formula (6), the eigenvalue of the one-fermion state is defined as

$$E_N=\lim_{\alpha\to0}\frac{\langle N\,|\,HS^\alpha(0,-\infty)\,|\,N\rangle}{\langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle}=m_0+\delta_N\Delta m_0+\delta E_N, \qquad (B.1)$$

$$\delta E_N=\lim_{\alpha\to0}\frac{\langle N\,|\,g(\psi^+\tau_1\psi)\,\hat{\varphi}(0)\,S^\alpha(0,-\infty)\,|\,N\rangle}{\langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle}. \qquad (B.1')$$

The matrix element in the numerator is

$$M_1^\alpha=\langle N\,|\,g(\psi^+\tau_1\psi)\,\hat{\varphi}(0)\,S^\alpha(0,-\infty)\,|\,N\rangle$$
$$=-\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q\int\limits_{-\infty}^{0}d\xi_1\ldots\int\limits_{-\infty}^{\xi_{q-1}}d\xi_q$$
$$\times\exp\Bigl(\alpha\sum_{j=1}^{q}\xi_j\Bigr)\,ig^2\int\limits_{-\infty}^{0}ds\,e^{\alpha s}i\Delta(s)$$
$$\times\prod_{j=1}^{q}\varepsilon(s-\xi_j)\exp\Bigl\{-\frac{ig^2}{2}\iint\limits_{-\infty}^{0}ds_1ds_2e^{\alpha(s_1+s_2)}$$
$$\times\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Bigr\}. \qquad (B.2)$$

The matrix element in the denominator of (B.1′) is obtained analogously

$$M_2^\alpha=\langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle=\sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q r_q,$$
$$r_q=\int\limits_{-\infty}^{0}d\xi_1\ldots\int\limits_{-\infty}^{\xi_{q-1}}d\xi_q\exp\Bigl(\alpha\sum_{j=1}^{q}\xi_j\Bigr)$$
$$\times\exp\Bigl\{-\frac{ig^2}{2}\iint\limits_{-\infty}^{0}ds_1ds_2e^{\alpha(s_1+s_2)}$$
$$\times\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Bigr\}. \qquad (B.3)$$

The integral in the exponential function is (when $\xi_1>\xi_2>\ldots>\xi$)

$$I_q(\xi_1,\ldots,\xi_q)$$
$$=-\frac{ig^2}{2}\iint\limits_{-\infty}^{0}ds_1ds_2e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)$$
$$\times\varepsilon(s_2-\xi_j)=-\frac{g^2}{4}\sum_{\mathbf{k}}\frac{1}{\omega^2}\Bigl(\frac{1}{i\alpha}+\frac{1}{\omega}\Bigr)-g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}$$
$$\times\Bigl[q+\sum_{l=1}^{q}(-1)^l e^{i\omega\xi_l}+2\sum_{l=2}^{q}\sum_{m=1}^{l-1}(-1)^{l+m}e^{-i\omega(\xi_m-\xi_l)}\Bigr]. \qquad (B.4)$$

The first term in (B.4) is the same for all the terms of the series, both in the numerator and in the denominator, and therefore cancels out. Subtracting in (B.2) the integral with respect to s, we obtain

$$M_1^\alpha=-\frac{g^2}{2}\sum_{\mathbf{k}}\omega^{-2}M_2^\alpha$$
$$-\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q\int\limits_{-\infty}^{0}d\xi_1\ldots\int\limits_{-\infty}^{\xi_{q-1}}d\xi_q\exp\Bigl(\alpha\sum_{j=1}^{q}\xi_j\Bigr)$$
$$\times i\Bigl(\frac{\partial}{\partial\xi_1}+\ldots+\frac{\partial}{\partial\xi_q}\Bigr)\exp\{I_q(\xi_1,\ldots,\xi_q)\}. \qquad (B.5)$$

Let us consider now the q-th term of the series (B.5)

$$\int\limits_{-\infty}^{0}d\xi_1\ldots\int\limits_{-\infty}^{\xi_{q-1}}d\xi_q\exp\Bigl(\alpha\sum_{j=1}^{q}\xi_j\Bigr)i\Bigl(\frac{\partial}{\partial\xi_1}+\ldots+\frac{\partial}{\partial\xi_q}\Bigr)$$
$$\times\exp\{I_q(\xi_1,\ldots,\xi_q)\}=ir_{q-1}-i\alpha q r_q. \qquad (B.6)$$

Further, substituting (B.6) in (B.5) we obtain for E_N, according to (A.1), the expression

$$E_N=m_0-\frac{g^2}{2}\sum_{\mathbf{k}}\frac{1}{\omega^2}+\lim_{\alpha\to0}i\alpha\frac{\sum\limits_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q q r_q}{\sum\limits_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q r_q}. \qquad (B.7)$$

Performing in (B.7) a term by term division of one series into another, we eliminate thereby the phase-shift factor in each power of Δm. This makes it possible to make the transition to the limit in α in each power with respect to Δm (see Appendix A).

As a result we obtain

$$E_N=m_0-\frac{1}{2}g^2\sum_{\mathbf{k}}\frac{1}{\omega^2}+\sum_{q=1}^{\infty}(-i\delta_N\Delta m)^q$$
$$\times\int\limits_0^{\infty}dx_1\ldots\int\limits_0^{\infty}dx_{q-1}\prod_{l=1}^{q-1}(1-\hat{Q}_l)F_{q-1}(x_1,\ldots,x_{q-1}), \qquad (B.8)$$

where F_{q-1} is given by (A.5). We obtain (6) directly from (B.8), if we take (A.9) into account.

APPENDIX C

Let us consider formula (13). We used the relations

$$\prod_{j=1}^{q}(1-\hat{Q}_j)=\sum_{l=0}^{q}(-1)^l\sum_{s_1=l}^{q}\ldots\sum_{s_i=l-i+1}^{s_{i-1}-1}\ldots\sum_{s_l=1}^{s_{l-1}-1}\hat{Q}_{s_1}\ldots\hat{Q}_{s_i}\ldots\hat{Q}_{s_l},\qquad (C.1)$$

$$\int_0^\infty dx_1\ldots\int_0^\infty dx_q e^{-\varepsilon(x_1+\ldots+x_q)}\times\hat{Q}_{s_1}\ldots\hat{Q}_{s_i}\ldots\hat{Q}_{s_l}F_q(x_1,\ldots,x_q)$$
$$=\left(\frac{1}{\varepsilon}\right)^l I^\varepsilon_{s_l-1}I^\varepsilon_{s_{l-1}-s_l-1}\ldots I^\varepsilon_{s_i-s_{i+1}-1}\ldots I^\varepsilon_{s_1-s_2-1}I^\varepsilon_{q-s_1},\qquad (C.2)$$

$$I^\varepsilon_s=\int_0^\infty dx_1\ldots\int_0^\infty dx_s e^{-\varepsilon(x_1+\ldots+x_s)}F_s(x_1,\ldots,x_s).\qquad (C.3)$$

Substituting these relations in (13), we obtain after interchanging the order of summation in the sums with respect to q, *l*, and s_i, and interchanging the summation indices (the derivations are simple but somewhat cumbersome and will therefore not be given here)

$$\delta m^\varepsilon=\sum_{q=0}^{\infty}(\Delta m)^q\bar{I}^\varepsilon_q=\sum_{l=0}^{\infty}\frac{(-\Delta m)^l}{\varepsilon^l}\left\{\sum_{q=0}^{\infty}(\Delta m)^q I^\varepsilon_q\right\}^{l+1},$$

$$\bar{I}^\varepsilon_q=\int_0^\infty dx_1\ldots\int_0^\infty dx_q e^{-\varepsilon(x_1+\ldots+x_q)}\times\prod_{j=1}^{q}(1-\hat{Q}_j)F_q(x_1,\ldots,x_q).\qquad (C.4)$$

Let us carry out in (C.4) formally the summation

$$\delta m^\varepsilon=\left[\frac{\Delta m}{\varepsilon}+\left\{\sum_{q=0}^{\infty}(\Delta m)^q I^\varepsilon_q\right\}^{-1}\right]^{-1}.\qquad (C.5)$$

Since $I^\epsilon_q \le (I^\epsilon_1)^q$, we get

$$\delta m^\varepsilon\leqslant[\Delta m/\varepsilon+1-\Delta m I^\varepsilon_1]^{-1}=[1-\Delta m\bar{I}^\varepsilon_1]^{-1}.\qquad (C.6)$$

The summation in (C.4) will be correct if ϵ satisfies the conditions

$$\Delta m I^\varepsilon_1<1,\qquad \Delta m/\varepsilon<1-\Delta m I^\varepsilon_1,\qquad (C.7)$$

of which the second is the stronger.

APPENDIX D

Let us consider the q-th term of the series (1′) for the Green's function

$$J_q(t)=(-i\Delta m)^q\int_0^t d\xi_1\ldots\int_0^{\xi_{q-1}}d\xi_q\exp\left\{-g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\left[\sum_{l=1}^{q}(-1)^l\left(e^{-i\omega(t-\xi_l)}-e^{-i\omega\xi_l}\right)+2\sum_{l=2}^{q}\sum_{m=1}^{l-1}(-1)^{l+m}e^{-i\omega(\xi_m-\xi_l)}\right]\right\}.\qquad (D.1)$$

Since we are interested in the behavior of the Green's function for small times ($\mu t \ll 1$), the functions in the exponential function can be replaced by their asymptotic expansions for small arguments

$$g^2\sum_{\mathbf{k}}\omega^{-3}e^{-i\omega\xi}=-G^2\ln(i\mu\xi)+G^2(\ln 2-C-1)+O(\mu\xi),\qquad (D.2)$$

where C is Euler's constant and $G^2 = g^2/2\pi^2$.

Making further a change of variables of integration in (D.1), $\xi_j = \mu t x_j$ and substituting the asymptotic expansion (D.2) therein, we obtain

$$J_q(t)=\left(-\frac{\Delta m}{\mu}\right)^q(i\mu t)^{q-2G^2[q/2]}\exp\{2G^2(\ln 2-C-1)[q/2]\}I_q,$$

$$I_q=\int_0^1 dx_1\ldots\int_0^{x_{q-1}}dx_q\prod_{r=1}^{q}[(1-x_r)/x_r]^{(-1)^r G^2}\times\prod_{l=2}^{q}\prod_{m=1}^{l-1}(x_m-x_l)^{(-1)^{l+m}2G^2},$$

$$\left[\frac{q}{2}\right]=\begin{cases}q/2 & \text{if q is even}\\(q-1)/2, & \text{if q is odd}\end{cases}\qquad (D.3)$$

Let us consider first the numbers I_q for even values q = 2n:

$$I_{2n}=\int_0^1 dx_1\ldots\int_0^{x_{2n-1}}dx_{2n}\prod_{r=1}^{n}\left[\frac{(1-x_{2r})x_{2r-1}}{x_{2r}(1-x_{2r-1})}\right]^{G^2}\times\prod_{l=1}^{n}(x_{2l-1}-x_{2l})^{-2G^2}\prod_{m=1}^{l-1}\left[\frac{(x_{2m}-x_{2l})(x_{2m-1}-x_{2l-1})}{(x_{2m}-x_{2l-1})(x_{2m-1}-x_{2l})}\right]^{2G^2}\qquad (D.4)$$

We make the following change of variables in the integrals:

$$x_l=\prod_{j=1}^{l}z_j.\qquad (D.5)$$

We then obtain

$$I_{2n}=\int_0^1 dz_1\ldots\int_0^1 dz_{2n}\prod_{s=1}^{2n}z_s^{(2n+1-s)(1-G^2)-1}\times\prod_{l=1}^{n}\left(\frac{1-\prod_{j=1}^{2l}z_j}{1-\prod_{j=1}^{2l-1}z_j}\right)^{G^2}(1-z_{2l})^{-2G^2}$$
$$\times\prod_{m=1}^{l-1}\left[\frac{\left(1-\prod_{j=2m+1}^{2l}z_j\right)\left(1-\prod_{j=2m}^{2l-1}z_j\right)}{\left(1-\prod_{j=2m+1}^{2l-1}z_j\right)\left(1-\prod_{j=2m}^{2l}z_j\right)}\right]^{2G^2}.\qquad (D.6)$$

It is easy to see that the integrals converge on the upper limit provided

$$2G^2 < 1, \qquad g^2/\pi^2 < 1. \tag{D.7}$$

To obtain an estimated upper limit for the number I_{2n}, we use the inequality

$$(1 - x)/(1 - x\beta) \leqslant 1, \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant \beta \leqslant 1. \tag{D.8}$$

An estimate yields

$$I_{2n} \leqslant \int_0^1 \frac{dz_1 z_1^{2n(1-G^2)-1}}{(1-z_1)^{G^2}} \prod_{l=2}^{2n} \int_0^1 dz_l \frac{z_l^{(2n+1-l)(1-G^2)-1}}{(1-z_l)^{2G^2}}. \tag{D.9}$$

The calculation of the integrals leads to the result

$$I_{2n} \leqslant \frac{[(1-G^2)\Gamma(1-2G^2)]^{2n}}{\Gamma((1-2G^2)/(1-G^2))} \Gamma\left(\frac{(2n+1)(1-G^2)-1}{1-G^2}\right)$$

$$\times \left[\frac{\Gamma(1-G^2)}{\Gamma((2n+1)(1-G^2))}\right]^2. \tag{D.10}$$

As n tends to infinity, the inequality assumes the form $[(2n+1)(1-2G^2) \gg 1]$

$$I_{2n} \leqslant \frac{[\Gamma(1-2G^2)]^{2n}}{\Gamma((2n+1)(1-2G^2))} \frac{(1-G^2)^{2n}[\Gamma(1-G^2)]^2}{\Gamma((1-2G^2)/(1-G^2))}$$

$$\times \left[\frac{(1-2G^2)^{1-2G^2}}{(1-G^2)^{2(1-G^2)}}\right]^{2n+1}. \tag{D.11}$$

The presence of the factor $\Gamma((2n+1)(1-2G^2))$ in the denominator of the right half of the inequality ensures the absolute convergence of the series (D.1) for small t, i.e., $\mu t \ll 1$, for arbitrary Δm, subject to the condition $g^2/\pi^2 < 1$ on the bare coupling constant. An estimate of the odd numbers I_{2n+1} is obtained analogously and yields the same results.

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A METHOD TO VERIFY EXPERIMENTALLY THAT THE SPEED OF LIGHT IS INDEPENDENT OF THE VELOCITY OF THE SOURCE

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A laboratory experiment is suggested which can be employed to verify directly whether the speed of light is independent of the velocity of the source. The effect is of the first order in v/c.

A direct experimental confirmation of the law that the speed of light does not depend on the velocity of the source would be a significant matter of principle.

The work published by A. M. Bonch-Bruevich[1,2] in 1956 is based on astronomical observations, namely comparison of the speed of light from different regions of the solar disc. We do not consider this work entirely convincing.

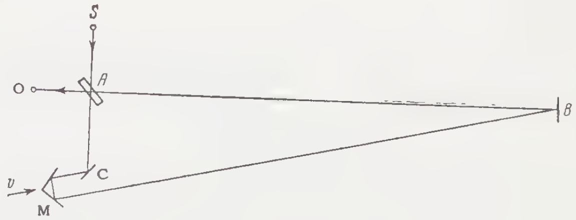
On the one hand, the author begins with some a priori assumptions concerning the speed of light after reflection from a mirror; on the other hand, owing to the low accuracy inherent in the method, the observed results have a large scatter, several times larger than the effect expected from classical theory. To obtain the final result the author had to treat statistically a very large number (1727) of observations, in which very large deviations were excluded outright.

In view of what has been said this work should not be regarded as a direct experimental verification of the independence of the speed of light of the source velocity.

Below we propose a scheme by which to verify directly in the laboratory the postulate of the constancy of the speed of light. In this method the difference between the effects predicted by the postulate and by classical theory is of the order v/c (not v^2/c^2), so that the postulate can be verified with an accuracy far greater than by any other method.

The basic idea of the method is to use an interferometer with a moving mirror. Calculation shows that the speed of the mirror in this experiment can be of the order of one meter per second and even less.

One variant of the experiment is shown in the figure. Here S is a monochromatic light source, A is a half-silvered mirror, B and C are plane mirrors, O is the observer of the interference pattern, and M is a system of two mutually perpendicular plane mirrors which can be moved in the direction of the arrow with a controlled speed v.



The system M will serve to keep the incident and reflected light parallel when the mirrors are moved, and to reduce greatly the influence of deviations from exact parallel motion. This system also eliminates the aberration of light, since here the aberration of one mirror is compensated by that of the other.

The beam traveling from S to A is split into two beams. The left beam follows the path ACMBA, and the right beam the path ABMCA. The two beams are recombined at A and both travel towards the observer O. If M remains fixed, both path lengths are the same, and the observer sees some definite interference pattern.

If the system of mirrors M is given a speed v, the two path lengths become unequal, and if the speed of light is constant, a shift of the interference fringes results. We shall show that the shift will be stable for any constant speed v.

We note first that the frequency ν is changed after reflection from M by the same amount for both beams; they return to A and arrive at O with the same frequency ν'.

Denote the length ACM by l_1, and the length MBA by l_2. Although l_1 and l_2 change, the difference $L = l_2 - l_1$ is constant. The left beam takes a time $t_1 = (l_1 - vt_1)/c$ to travel through ACM, so that $t_1 = l_1/(c+v)$. The same beam takes a time $t_2 = (l_2 - vt_1)/c = [l_2 - vl_1/(c+v)]/c$ to cover the distance MBA. The total time spent by the left beam is

$$T_L = t_1 + t_2 = \frac{l_1}{c+v} + \frac{1}{c}\left(l_2 - \frac{vl_1}{c+v}\right).$$

Similarly the total time for the right beam is

$$T_R = \frac{l_2}{c+v} + \frac{1}{c}\left(l_1 - \frac{vl_2}{c+v}\right).$$

The difference in time is

$$\Delta T = T_L - T_R = \frac{2v\,(l_2 - l_1)}{c\,(c+v)} = \text{const}$$

Therefore the interference pattern is stable.

The fringe shift is

$$\delta = c\Delta T/\lambda' = 2vL/\lambda'(c+v) \approx 2vL/\lambda c. \qquad (1)$$

For example, for $L = 30$ m, $v = 1$ m/sec and $\lambda \sim 5000$ A, the pattern is shifted by 0.4 fringes.

If the speed of light depended on the velocity of the source, it would have the value $c + 2v$ after reflection from M. This would be exactly compensated by the difference in path lengths and there would be no fringe shift:

$$T_L = \frac{l_1}{c+v} + \frac{1}{c+2v}\left(l_2 - \frac{vl_1}{c+v}\right) = \frac{l_1 + l_2}{c+2v} = T_R$$

Therefore the experiment suggested by us can show with high accuracy that the speed of light is independent of the velocity of the source.

Small deviations from parallelism and uniformity in the motion of the mirror, small errors in the length L and the speed v, performance of the experiment in air instead of vacuum, etc., deviations which in our estimate will not exceed a few percent, cannot spoil the outcome of the experiment. If it should turn out for some unforeseen reason that the actual errors exceed our estimate considerably, or if the experimenter has only an interferometer of low sensitivity available, the fringe shift could be increased by increasing L or v.

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[1] A. M. Bonch-Bruevich, Doklady Akad. Nauk SSSR **109**, 481 (1956), Soviet Phys.-Doklady **1**, 430 (1956).

[2] A. M. Bonch-Bruevich and V. A. Molchanov, Оптика и спектроскопия (Optics and Spectroscopy) **1**, 113 (1956).

Translated by D. R. Brill
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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

ACCURACY OF ADIABATIC INVARIANT OF A PARTICLE IN A HIGH-DENSITY PLASMA

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An expression is obtained for the variation of the adiabatic invariant in reflection of a charged particle from a magnetic mirror; the diamagnetism of the plasma is taken into account. The limiting case of complete expulsion of the magnetic field from the plasma, corresponding to high plasma densities, is considered.

THE variation of the adiabatic invariant of a single particle in a magnetic field has been found in reference 1. The results obtained in this work apply for low plasma densities in a mirror system, where the self magnetic field of the plasma can be neglected. It is of interest to consider the effect of the self magnetic field of the plasma on the motion of a single particle. An accurate analysis of this problem is extremely complicated because it is necessary to treat the motion of the particle in a self-consistent field. For this reason we consider the limiting case of high density and temperature.

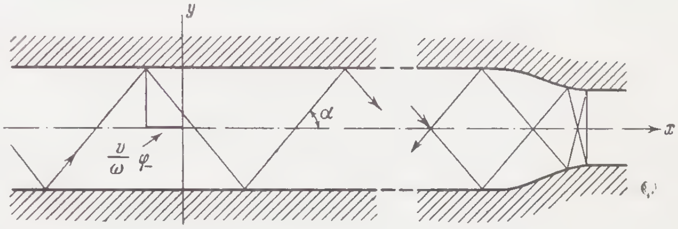
Because of the high density, the magnetic field is completely expelled from the region occupied by the plasma. We assume that the magnetic field does not enter the region occupied by the plasma and that it acts as a wall from which the particles undergo specular reflection.* Because the temperature is high the variation in the adiabatic invariant due to collisions can be neglected. A model of the magnetic-mirror system being considered is shown in the figure. The unhatched region is occupied by plasma while the hatched region is occupied by the magnetic field. For simplicity we consider the plane problem. The extension to the axially symmetric case does not involve any fundamental difficulties.

Let the equation of the curves which bound the region occupied by the plasma be of the form

$$y = \pm f(x). \tag{1}$$

The broken line in the figure represents a portion of the particle trajectory. The adiabatic invariant is given by the expression

$$I = \frac{1}{2\pi}\oint p_y dy = 2\pi^{-1} mvf(x)\sin\alpha(x).$$



Consequently, the adiabatic invariant suffers some variation when the particle is reflected from the magnetic mirror because the angle α is changed. We now calculate this variation.

There are certain difficulties in the classical formulation of this problem (compared with the motion of a particle in a magnetic field) because even in the zeroth approximation the equation for the transverse motion is nonlinear. We shall use a method similar to that developed in reference 2. Specifically, we first solve the quantum-mechanical problem and then take the limits which correspond to the classical case. In the quantum-theoretical formulation of the problem the difficulty noted above is unimportant, as will be apparent from the behavior of the solution.

To solve the quantum-mechanical problem of the motion of a particle in a potential well of the present configuration we use the same coordinate system used in reference 2. The new variables ξ and η are expressed in terms of x and y by means of the equations

$$\eta = \frac{y}{f(x)}, \qquad \int\limits_x^\xi \frac{f(t)}{f'(t)}\,dt = y^2. \tag{2}$$

It can easily be shown that the adiabaticity condition for the problem is $f'(x) \ll 1$. Assuming that f′ is small, we write the Laplacian in the coordinates ξ and η, keeping second-order terms only:

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*The possibility of this formulation of the problem has been called to the attention of the author by B. V. Chirikov.

$$L = L_0 + L_1, \qquad L_0 = -\frac{1}{f^2(\xi)}\frac{\partial^2}{\partial\eta^2} - \frac{1}{f(\xi)}\frac{\partial}{\partial\xi}\left(f(\xi)\frac{\partial}{\partial\xi}\right),$$

$$L_1 = \eta^2\left[f'^2 + \frac{1}{2}f''f\right]L_0 + \frac{1}{f^2}\left[f'^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right)$$
$$-\frac{\eta^2}{f}\frac{\partial}{\partial\xi}\left\{f\left[f'^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\xi}\right\}. \tag{3}$$

For our purposes it will be necessary to solve the Schrödinger equation with the boundary conditions $\psi = 0$ for $\eta = \pm 1$. The zeroth-approximation equation is of the form

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{f^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{f}\frac{\partial}{\partial\xi}\left(f\frac{\partial}{\partial\xi}\right)\right\}\psi = E\psi. \tag{4}$$

Separating variables in Eq. (4) we obtain a family of wave functions normalized for a δ-function energy:

$$\psi_{nE} = Z_{nE}\sin\lambda_n\eta, \qquad \psi_{n+1/2,\,E} = Z_{n+1/2,\,E}\cos\lambda_{n+1/2}\eta;$$

$$Z_{nE} = \left(\frac{m}{2\pi k_{nE}\hbar^2 f}\right)^{1/3}\exp\left\{i\int^{\xi} k_{nE}\,d\xi\right\},$$

$$k_{nE}^2 = 2mE/\hbar^2 - \lambda_n^2/f^2, \; \lambda_n = \pi n. \tag{5}$$

For our purpose it is necessary to compute the transition amplitudes between different states (5) under the effect of the perturbation. These amplitudes must be such that the squares of their moduli equal a dimensionless transition probability which represents the transition probability for the case of single passage of a particle through an inhomogenity (see Landau and Lifshitz)[3]. In order to obtain this transition amplitude the original wave function must be normalized for unit flux while the final wave function must be normalized for a delta-function energy. With these requirements in mind we compute the matrix elements for the perturbation operator between states (4).

We calculate only near-diagonal elements of the scattering matrix, which are of greatest magnitude. The matrix elements for transitions between states with different parity in η vanish as a consequence of the parity of the perturbation. In calculating the matrix elements we assume that the greatest contribution in the integral over ξ is due to a simple pole of the function $f(\xi)$ so that

$$a_{n,n-1} = \frac{4\pi I}{3\hbar}\, i\exp\left\{2i\int^{\xi_0}\frac{\omega}{v}\,d\xi\right\}; \qquad a_{n,n+1} = -a^*_{n,n-1}. \tag{6}$$

Here, $\omega = \pi^2 I/2mf^2$ is the frequency of the classical transverse motion and $v = [2(E - I\omega)m]^{1/2}$ is the velocity of the longitudinal motion. In a way similar to that used in reference 2 it can be shown that the higher-order perturbations make only a small contribution compared with the first-order perturbation.

To make the transition to the classical limit we form, following reference 2, a wave packet which describes the particle moving along the trajectory. To describe the trajectory and particle uniquely it is necessary to assign the adiabatic invariant I_- and the phase φ_- as $x \to -\infty$. This phase must be introduced into the wave functions in such a way that the resulting wave packet actually describes a classical particle with the given phase. This procedure makes it possible to avoid the ambiguity in Eq. (6) which results from the fact that there is no lower limit of integration.

It can be shown[4] that the integral in Eq. (6) assumes the following form when the ambiguity noted above is removed:

$$\int^{x}\frac{\omega}{v}\,dx = \int_{-\infty}^{x}\left(\frac{\omega}{v} - \frac{\omega_-}{v_-}\right)dx + \frac{\omega_-}{v_-}x + \varphi_-. \tag{7}$$

The minus subscript indicates that a given quantity is taken as $x \to -\infty$. The meaning of the quantity φ_- is indicated in the figure.

The classical adiabatic invariant is computed as the average over the packet of the quantity $\hbar(n + 1/2)$, where n is the transverse quantum number. Carrying out this averaging process as in reference 2, we obtain the variation in the adiabatic invariant:

$$\frac{I_+ - I_-}{I} = \frac{8\pi}{3}\,\mathrm{Re}\left[i\exp\left(2i\int^{\xi_0}\frac{\omega}{v}\,d\xi\right)\right]. \tag{8}$$

We see that the variation of the invariant is exponentially small in the case being considered, as it is for the motion of a single particle in a magnetic field.

When the region occupied by the plasma is asymmetrical with respect to the x axis, we can show by analogy with reference 1 that the expression for the variation of the adiabatic invariant differs from that given in Eq. (8): there is no factor of 2 in the exponential and the factor which multiplies the exponential is different.

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[2] A. M. Dykhne and V. L. Pokrovskii, JETP **39**, 373 (1960), Soviet Phys. JETP **12**, 264 (1960).

[3] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Pergamon, 1958.

[4] A. M. Dykhne. Dissertation, Institute for Radiophysics and Electronics, Siberian Branch, Academy of Sciences, U.S.S.R., 1960.

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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

METHOD OF SUCCESSIVE EXTENSION OF THE SPECTRAL FUNCTIONS IN THE MANDELSTAM REPRESENTATION

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A method is developed which makes it possible to extend the range of knowledge of the spectral functions and absorption parts, by a series of successive steps, from a knowledge of the absorption part of the scattering amplitudes specified in the physical region. By means of this method, the NN scattering amplitude can be expressed in terms of the $\pi\pi$ and πN scattering amplitudes, specified only in the physical regions.

ALTHOUGH representations of the Mandelstam type for the scattering amplitude have not been proved, it is quite probable that they exist. These representations, together with the conditions of unitarity in the two-particle representation, are employed by us in the present work as the basis for an approximate calculation of the amplitude of nucleon-nucleon scattering.

In the two-particle approximation to the unitarity condition, the amplitude of NN scattering is connected with the amplitudes of $\pi\pi$ and πN scattering. The purpose of our research is the development of a semiphenomenological method of constructing the NN scattering amplitude from the values of the $\pi\pi$ and πN amplitudes in the physical regions.

The work consists of two parts. In Part I a summary is given of Mandelstam's equations[1,2] for all possible channels of $\pi\pi$, πN and NN scattering. For simplicity, the spin-charge variables are neglected and subtraction is not considered.

In Part II it is shown how one can systematically find the spectral functions of the Mandelstam representation in ever wider regions according to the absorption parts of the amplitude given in the physical regions. It is very important that these regions grow quickly and that the spectral functions can be computed exactly in that part of them in which it is not necessary to take into account the graphs of perturbation theory corresponding to inelastic processes over two channels. It is further shown how one can express the amplitude of NN scattering in terms of the absorption parts of the $\pi\pi$ and πN scattering amplitudes in the physical regions, and in terms of the absorption part of the amplitude of nucleon-antinucleon annihilation into two π mesons. The latter can be expressed in terms of the first two absorption parts.[3,4]

I. THE MANDELSTAM EQUATIONS FOR THE SPECTRAL FUNCTIONS

1. Scattering of Pions on Pions

Let us consider the reactions

$$\pi(q_1) + \pi(q_2) \rightarrow \pi(q_3) + \pi(q_4), \tag{A.I}$$

$$\pi(q_1) + \pi(-q_4) \rightarrow \pi(q_3) + \pi(-q_2), \tag{A.II}$$

$$\pi(q_1) + \pi(-q_3) \rightarrow \pi(-q_2) + \pi(q_4) \tag{A.III}$$

[$\pi(q)$ denotes a π meson with four-dimensional momentum q] and introduce the relativistically invariant quantities [the squares of the energies in the center of mass system (c.m.s.) for these three reactions]

$$\sigma_1 = (q_1 + q_2)^2, \quad \sigma_2 = (q_1 - q_4)^2, \quad \sigma_3 = (q_1 - q_3)^2, \tag{1}$$

which obviously satisfies the relation

$$\sigma_1 + \sigma_2 + \sigma_3 = 4\mu^2, \tag{2}$$

where μ — rest mass of the π meson.

We write down the double spectral representation for the amplitude A of the reactions considered here:

$$A = \frac{1}{\pi^2} \sum_{1 \leqslant i < k}^{3} \int_{4\mu^2}^{\infty} d\sigma_i' \int_{4\mu^2}^{\infty} d\sigma_k' \frac{A_{ik}(\sigma_i', \sigma_k')}{(\sigma_i' - \sigma_i)(\sigma_k' - \sigma_k)}, \tag{3}$$

where the real spectral functions $A_{ik}(x, y)$ ($1 \le i < k \le 3$) satisfy the relations*

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*Actually, and this will be proved in Sec. 4, the functions $A_{ik}(x, y)$ are equal to zero in a much broader region than shown in the limits of integration of (3).

$$A_{12}(x, y) = A_{13}(x, y) = A_{23}(x, y); \tag{4}$$

$$A_{ik}(x, y) = A_{ik}(y, x) \qquad (1 \leqslant i < k \leqslant 3). \tag{5}$$

Furthermore, let us introduce the absorption parts of the amplitude

$$A_1(\sigma_1, \sigma_3) = \frac{1}{\pi}\int d\sigma_2' \frac{A_{12}(\sigma_1, \sigma_2')}{\sigma_2' - \sigma_2} + \frac{1}{\pi}\int d\sigma_3' \frac{A_{13}(\sigma_1, \sigma_3')}{\sigma_3' - \sigma_3}, \tag{6}$$

$$A_2(\sigma_2, \sigma_3) = \frac{1}{\pi}\int d\sigma_1' \frac{A_{12}(\sigma_1', \sigma_2)}{\sigma_1' - \sigma_1} + \frac{1}{\pi}\int d\sigma_3' \frac{A_{23}(\sigma_2, \sigma_3')}{\sigma_3' - \sigma_3}, \tag{7}$$

$$A_3(\sigma_3, \sigma_1) = \frac{1}{\pi}\int d\sigma_1' \frac{A_{13}(\sigma_1', \sigma_3)}{\sigma_1' - \sigma_1} + \frac{1}{\pi}\int d\sigma_2' \frac{A_{23}(\sigma_2', \sigma_3)}{\sigma_2' - \sigma_2}, \tag{8}$$

where the integration in the expressions for $A_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration, at which the corresponding spectral function differs from zero.

It is clear that

$$A_i(x, y) = 0, \quad \text{if} \quad x < 4\mu^2, \tag{9}$$

$$A_1(x, y) = A_2(x, y) = A_3(x, y), \tag{10}$$

$$A_i(x, y) = A_i(x, 4\mu^2 - x - y). \tag{11}$$

For derivation of the set of Mandelstam equations, let us write the unitarity condition in the approximation of elastic scattering, for example, for the reaction (A.III):*

$$A_{13}^{(3)}(\xi, \eta) = -\frac{1}{4\pi^2\eta^{1/2}(\eta/4 - \mu^2)^{1/2}} \int\limits_{(\xi_1 < \xi)} dxdy \frac{A_1^*(x, \eta)\, A_1(y, \eta)}{[(\xi - \xi_1)(\xi - \xi_2)]^{1/2}}, \tag{12}$$

$$\xi_{1,2} = \xi_{1,2}(\eta; x, y) = x + y + \frac{2xy}{\eta - 4\mu^2} \pm \frac{2}{\eta - 4\mu^2}$$

$$\times [x^2 + (\eta - 4\mu^2)\, x]^{1/2} [y^2 + (\eta - 4\mu^2)\, y]^{1/2}, \tag{13}$$

and integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude $A_{13}^{(3)}(\sigma_1, \sigma_3)$ represents the part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs corresponding to elastic scattering with energy σ_3 and inelastic scattering with energy σ_1 make a contribution, so that we have

$$A_{13}^{(3)}(\sigma_1, \sigma_3) = 0, \text{ if } \quad \sigma_1 < 16\mu^2 \text{ or } \sigma_3 < 4\mu^2. \tag{14}$$

Introducing the notation

$$A_{13}^{(3)}(\sigma_1, \sigma_3) = \chi(\sigma_1, \sigma_3), \tag{15}$$

we have, successively,

$$A_{13}(\sigma_1, \sigma_3) = \chi(\sigma_1, \sigma_3) + \chi(\sigma_3, \sigma_1) + X_A(\sigma_1, \sigma_3), \tag{16}$$

where $\chi(\sigma_3, \sigma_1)$ is the part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs corresponding to elastic scattering with energy σ_1, and inelastic scattering with energy σ_3 make a contribution, while $X_A(\sigma_1, \sigma_3)$ $[X_A(\sigma_1, \sigma_3) = X_A(\sigma_3, \sigma_1)]$ is that part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs make a contribution, corresponding to inelastic scattering both with energy σ_1 and with energy σ_3, so that, finally,

$$X_A(\sigma_1, \sigma_3) = 0, \text{ if } \quad \sigma_1 < 16\mu^2 \text{ or } \sigma_3 < 16\mu^2. \tag{17}$$

Then substituting (16) in (6) and taking (4) into account, we get

$$A_1(\sigma_1, \sigma_3) = \frac{1}{\pi}\int dx\, [\chi(x, \sigma_1)$$

$$+ \chi(\sigma_1, x)]\left(\frac{1}{x - \sigma_2} + \frac{1}{x - \sigma_3}\right) + A_1^{uu}(\sigma_1, \sigma_3), \tag{18}$$

where

$$A_1^{uu}(\sigma_1, \sigma_3) = \frac{1}{\pi}\int dx\, X_A(\sigma_1, x)\left(\frac{1}{x - \sigma_2} + \frac{1}{x - \sigma_3}\right). \tag{19}$$

Equations (12), (15), (18), and (19) represent the desired set of Maxwell's equations in the case of the scattering of π mesons on π mesons.

2. Scattering of Pions on Nucleons

Now consider the reactions

$$\pi(q_1) + N(p_1) \rightarrow \pi(q_2) + N(p_2), \tag{B.I}$$

$$\pi(-q_2) + N(p_1) \rightarrow \pi(-q_1) + N(p_2), \tag{B.II}$$

$$N(p_1) + \bar{N}(-p_2) \rightarrow \pi(-q_1) + \pi(q_2) \tag{B.III}$$

[N(p) and $\bar{\text{N}}$(p′) denote respectively a nucleon with four-dimensional momentum p and an antinucleon with four-dimensional momentum p′] and introduce the relativistically invariant quantities (the squares of the energies in the c.m.s. for these three reactions)

$$s = (q_1 + p_1)^2,$$

$$\bar{s} = (p_1 - q_2)^2, \qquad u = (p_1 - p_2)^2, \tag{20}$$

which obviously satisfy the relation

$$s + s + u = 2(m^2 + \mu^2), \tag{21}$$

where m — rest mass of the nucleon.

We write down the double spectral representation for the amplitude B of the reactions considered here:*

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*In view of the symmetry of the reactions of (A.I), (A.II), and (A.III) one need not consider the unitarity conditions for the first two reactions.

*Actually, as will be proved in Sec. 5, the functions B_{ik} (x, y) are equal to zero over a much wider region.

$$B = \frac{g^2}{m^2 - s} + \frac{g^2}{m^2 - \bar{s}} + \frac{1}{\pi^2} \int\limits_{(m+\mu)^2}^{\infty} ds' \int\limits_{(m+\mu)^2}^{\infty} d\bar{s}' \frac{B_{12}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})}$$

$$+ \frac{1}{\pi^2} \int\limits_{(m+\mu)^2}^{\infty} ds' \int\limits_{4\mu^2}^{\infty} du' \frac{B_{13}(s', u')}{(s' - s)(u' - u)}$$

$$+ \frac{1}{\pi^2} \int\limits_{(m+\mu)^2}^{\infty} d\bar{s}' \int\limits_{4\mu^2}^{\infty} du' \frac{B_{23}(\bar{s}', u)}{(\bar{s}' - \bar{s})(u' - u)}, \tag{22}$$

where g is the interaction constant of π mesons with nucleons, and the real spectral functions $B_{ik}(x, y)$ $(1 \le i < k \le 3)$ satisfy the relations

$$B_{12}(x, y) = B_{12}(y, x), \quad B_{13}(x, y) = B_{23}(x, y). \tag{23}$$

We further introduce the absorption parts of the amplitude

$$B_1(s, u) = \frac{1}{\pi} \int d\bar{s}' \frac{B_{12}(s, \bar{s}')}{\bar{s}' - \bar{s}} + \frac{1}{\pi} \int du' \frac{B_{13}(s, u')}{u' - u}, \tag{24}$$

$$B_2(\bar{s}, u) = \frac{1}{\pi} \int ds' \frac{B_{12}(s', \bar{s})}{s' - s} + \frac{1}{\pi} \int du' \frac{B_{23}(\bar{s}, u')}{u' - u}, \tag{25}$$

$$B_3(u, s) = \frac{1}{\pi} \int ds' \frac{B_{13}(s', u)}{s' - s} + \frac{1}{\pi} \int d\bar{s}' \frac{B_{23}(\bar{s}', u)}{\bar{s}' - \bar{s}}, \tag{26}$$

where integration in the expressions for $B_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration for which the corresponding spectral function differs from zero.

It is clear that

$$B_i(x, y) = 0 \text{ for } x < (m+\mu)^2 \ (i = 1, 2);$$

$$B_3(x, y) = 0, \text{ for } x < 4\mu^2; \tag{27}$$

$$B_1(x, y) = B_2(x, y),$$

$$B_3(x, y) = B_3(x, 2(m^2 + \mu^2) - x - y). \tag{28}$$

For derivation of the set of Mandelstam equations, we write the condition for unitarity in the approximation of elastic scattering first for the reaction (B.I):

$$B_{13}^{(1)}(s, u) = -\frac{1}{8\pi^2 k \sqrt{s}} \left\{ \int\limits_{(u_1 < u)} dxdy \frac{B_3^*(x, s) B_3(y, s)}{[(u - u_1)(u - u_2)]^{1/2}} \right.$$

$$\left. + \int\limits_{(u_3 < u)} dxdy \frac{\widetilde{B}_2^*(x, u(s, x)) \widetilde{B}_2(y, u(s, y))}{[(u - u_3)(u - u_4)]^{1/2}} \right\}, \tag{29}$$

$$B_{12}^{(1)}(s, \bar{s}) = -\frac{1}{8\pi^2 k \sqrt{s}} \int\limits_{(\bar{s}_1 < \bar{s})} dxdy$$

$$\times \frac{\widetilde{B}_2^*(x, u(s, x)) B_3(y, s) + \widetilde{B}_2(x, u(s, x)) B_3^*(y, s)}{[(\bar{s} - \bar{s}_1)(\bar{s} - \bar{s}_2)]^{1/2}}, \tag{30}$$

where

$$u_{1,2} = u_{1,2}(s; x, y) = x + y + \frac{xy}{2k^2}$$

$$\pm \frac{1}{2k^2} (x^2 + 4k^2 x)^{1/2} (y^2 + 4k^2 y)^{1/2}, \tag{31}$$

$$u_{3,4} = u_{3,4}(s; x, y) = x + y + \frac{1}{2k^2} [xy - (m^2 - \mu^2)^2$$

$$+ (x + y - 2\alpha)(s - 4k^2 - 2\alpha) \pm \Delta(s, x) \Delta(s, y)], \tag{32}$$

$$\bar{s}_{1,2} = \bar{s}_{1,2}(s; x, y) = x + y + \frac{1}{2k^2} [xy + y(s - 4k^2 - 2\alpha)$$

$$\pm (y^2 + 4k^2 y)^{1/2} \Delta(s, x)], \tag{33}$$

$$\Delta(s, x) = [(\alpha - x)^2 - 2(s - 2k^2 - 2\alpha)(\alpha - x)$$

$$+ 4\alpha k^2 + 4m^2\mu^2]^{1/2}, \quad \alpha = m^2 + \mu^2, \tag{34}$$

$$\widetilde{B}_2(\bar{s}, u) = \pi g^2 \delta(\bar{s} - m^2) + B_2(\bar{s}, u),$$

$$u(s, \bar{s}) = 2(m^2 + \mu^2) - s - \bar{s}; \tag{35}$$

k is connected with s by the relation $s = [(k^2 + m^2)^{1/2} + (k^2 + \mu^2)^{1/2}]^2$, and the integration is carried out over all real x and y for which the corresponding absorption parts are different from zero.

$B_{13}^{(1)}(s, u)$ is the part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy s and inelastic scattering with energy u make a contribution, so that we have

$$B_{13}^{(1)}(s, u) = 0, \text{ if } s < (m + \mu)^2 \text{ or } u < 16\mu^2. \tag{36}$$

$B_{12}^{(1)}(s, \bar{s})$ is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to elastic scattering with energy s and inelastic scattering with energy $\bar{s}$ make a contribution, so that we have

$$B_{12}^{(1)}(s, \bar{s}) = 0, \text{ if } s < (m + \mu)^2 \text{ or } \bar{s} < (m + 2\mu)^2. \tag{37}$$

We now write down the unitarity condition in the approximation of elastic scattering for the reaction (B.III):*

$$B_{13}^{(3)}(s, u) = -\frac{1}{4\pi^2 u^{1/2} (u/4 - \mu^2)^{1/2}}$$

$$\times \int\limits_{(s_1 < s)} dxdy \frac{\widetilde{B}_1(x, u) A_1(y, u)}{[(s - s_1)(s - s_2)]^{1/2}}, \tag{38}$$

where

$$s_{1,2} = s_{1,2}(u; x, y) = x + y + \frac{2}{u - 4\mu^2} \{ y(x - m^2 + \mu^2)$$

$$\pm [(x - m^2 + \mu^2)^2 + (u - 4\mu^2)x]^{1/2} [y^2 + (u - 4\mu^2) y]^{1/2} \}; \tag{39}$$

$$\widetilde{B}_1(x, u) = \pi g^2 \delta(x - m^2) + B_1(x, u), \tag{40}$$

and the integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude of $B_{13}^{(3)}(s, u)$ is that part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy u and inelastic scattering with energy s make a contribution such that

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*In view of the symmetry of the reactions (B.I) and (B.II), we need not consider the unitarity condition for the reaction (B.II). We shall assume that the unitarity condition for the reaction (B.III) is valid also in the non-physical region of energy $4\mu^2 < u < 4m^2$.

$$B_{13}^{(3)}(s, u) = 0, \quad \text{if} \quad s < (m + 2\mu)^2 \text{ or } u < 4\mu^2. \qquad (41)$$

Introducing the notation

$$B_{13}^{(1)}(s, u) = \psi_1(s, u), \quad B_{12}^{(1)}(s, \bar{s}) = \psi_2(s, \bar{s}), \quad B_{13}^{(3)}(s, u) = \psi_3(s, u), \qquad (42)$$

we obtain

$$B_{13}(s, u) = \psi_1(s, u) + \psi_3(s, u) + X_B(s, u), \qquad (43)$$

$$B_{12}(s, \bar{s}) = \psi_2(s, \bar{s}) + \psi_2(\bar{s}, s) + Y_B(s, \bar{s}). \qquad (44)$$

Here $\psi_2(\bar{s}, s)$ is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to elastic scattering with energy $\bar{s}$ and inelastic scattering with energy s make a contribution; $X_B(s, u)$ is the part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy s and with energy u make a contribution; $Y_B(s, \bar{s})$ [$Y_B(s, \bar{s}) = Y_B(\bar{s}, s)$] is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy s and with energy $\bar{s}$ make a contribution, so that certainly

$$X_B(s, u) = 0, \quad \text{if} \quad s < (m + 2\mu)^2 \text{ or } u < 16\mu^2, \qquad (45)$$

$$Y_B(s, \bar{s}) = 0, \quad \text{if} \quad s < (m + 2\mu)^2 \text{ or } \bar{s} < (m + 2\mu)^2. \qquad (46)$$

Finally, substituting (43), (44) in (24), (26) and taking (23) into account, we obtain

$$B_1(s, u) = \frac{1}{\pi}\int d\bar{s}' \frac{\psi_2(s, \bar{s}') + \psi_2(\bar{s}', s)}{\bar{s}' - \bar{s}} + \frac{1}{\pi}\int du' \frac{\psi_1(s, u') + \psi_3(s, u')}{u' - u} + B_1^{uu}(s, u), \qquad (47)$$

$$B_3(u, s) = \frac{1}{\pi}\int dx\,[\psi_1(x, u) + \psi_3(x, u)]\left(\frac{1}{x - s} + \frac{1}{x - \bar{s}}\right) + B_3^{uu}(u, s), \qquad (48)$$

where

$$B_1^{uu}(s, u) = \frac{1}{\pi}\int d\bar{s}' \frac{Y_B(s, \bar{s}')}{\bar{s}' - \bar{s}} + \frac{1}{\pi}\int du' \frac{X_B(s, u')}{u' - u}, \qquad (49)$$

$$B_3^{uu}(u, s) = \frac{1}{\pi}\int dx X_B(x, u)\left(\frac{1}{x - s} + \frac{1}{x - \bar{s}}\right). \qquad (50)$$

Equations (29), (30), (38), and (47) — (50) are the desired set of Mandelstam equations in the case of the scattering of π mesons on nucleons.

3. Scattering of Nucleons on Nucleons

Finally, we shall consider the reactions

$$N(n_1) + N(p_1) \to N(n_2) + N(p_2), \qquad (\text{C.I})$$

$$N(n_1) + \bar{N}(-p_2) \to \bar{N}(-p_1) + N(n_2), \qquad (\text{C.II})$$

$$N(n_1) + \bar{N}(-n_2) \to \bar{N}(-p_1) + N(p_2) \qquad (\text{C.III})$$

and introduce the relativistically invariant quantities (the squares of the energy in the c.m.s. for the three reactions)

$$w = (n_1 + p_1)^2, \quad \bar{t} = (n_1 - p_2)^2, \quad t = (n_1 - n_2)^2, \qquad (51)$$

which obviously satisfy the relation

$$w + \bar{t} + t = 4m^2. \qquad (52)$$

We now write down the double spectral representation for the amplitude C of the reactions considered here:*

$$C = \frac{g^2}{\mu^2 - \bar{t}} + \frac{g^2}{\mu^2 - t} + \frac{1}{\pi^2}\int_{4m^2}^{\infty} dw' \int_{4\mu^2}^{\infty} d\bar{t}' + \frac{C_{12}(w', \bar{t}')}{(w' - w)(\bar{t}' - \bar{t})} + \frac{1}{\pi^2}\int_{4m^2}^{\infty} dw' \int_{4\mu^2}^{\infty} dt' \frac{C_{13}(w', t')}{(w' - w)(t' - t)} + \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} d\bar{t}' \int_{4\mu^2}^{\infty} dt' \frac{C_{23}(\bar{t}', t')}{(\bar{t}' - \bar{t})(t' - t)}, \qquad (53)$$

where the real spectral functions $C_{ik}(x, y)$ $(1 \leq i < k \leq 3)$ satisfy the relations

$$C_{12}(x, y) = C_{13}(x, y), \quad C_{23}(x, y) = C_{23}(y, x). \qquad (54)$$

We introduce the absorption parts of the amplitude

$$C_1(w, t) = \frac{1}{\pi}\int d\bar{t}' \frac{C_{12}(w, \bar{t}')}{\bar{t}' - \bar{t}} + \frac{1}{\pi}\int dt' \frac{C_{13}(w, t')}{t' - t}, \qquad (55)$$

$$C_2(\bar{t}, w) = \frac{1}{\pi}\int dw' \frac{C_{12}(w', t)}{w' - w} + \frac{1}{\pi}\int dt' \frac{C_{23}(\bar{t}, t')}{t' - t}, \qquad (56)$$

$$C_3(t, w) = \frac{1}{\pi}\int dw' \frac{C_{13}(w', t)}{w' - w} + \frac{1}{\pi}\int d\bar{t}' \frac{C_{23}(\bar{t}', t)}{\bar{t}' - \bar{t}}, \qquad (57)$$

where the integration in the expression for $C_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration for which the corresponding spectral function is different from zero.

It is clear that

$$C_1(x, y) = 0, \quad \text{if} \quad x < 4m^2; \quad C_i(x, y) = 0, \quad \text{or } x < 4\mu^2 \; (i = 2,3); \qquad (58)$$

$$C_1(x, y) = C_1(x, 4m^2 - x - y); \quad C_2(x, y) = C_3(x, y). \qquad (59)$$

For derivation of the set of Mandelstam equations, we write down the unitarity condition initially for the reaction (C.I) in the elastic scattering approximation:

$$C_{13}^{(1)}(w, t) = -\frac{1}{4\pi^2 w^{1/2}(w/4 - m^2)^{1/2}} \times \int_{(t_1 < t)} dxdy \frac{\widetilde{C}_3^*(x, w)\,\widetilde{C}_3(y, w)}{[(t - t_1)(t - t_2)]^{1/2}}, \qquad (60)$$

$$t_{1,2} = t_{1,2}(w; x, y) = x + y + \frac{2}{w - 4m^2}\{xy \pm [x^2 + x(w - 4m^2)]^{1/2} \times [y^2 + y(w - 4m^2)]^{1/2}\}, \qquad (61)$$

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*In reality, as will be shown in Sec. 6, the functions $C_{ik}(x, y)$ $(1 \leq i < k \leq 3)$ are equal to zero in a wider region than shown in (53).

$$\widetilde{C}_3(t, w) = \pi g^2 \delta(t - \mu^2) + C_3(t, w), \tag{62}$$

and the integration is carried out over all real x and y for which the absorption parts are different from zero.

$C_{13}^{(1)}(w, t)$ consists of two components:

$$C_{13}^{(1)}(w, t) = \Lambda(w, t) + \Phi_1(w, t). \tag{63}$$

The first component corresponds to the case in which both variables of integration are equal to μ^2. It is clear that

$$\Lambda(w,t) = -(g^4/4) w^{-1/2} (w/4 - m^2)^{-1/2} [t - t_1(w; \mu^2, \mu^2)]^{-1/2} \times [t - t_2(w; \mu^2, \mu^2)]^{-1/2} \tag{64}$$

for those w and t for which $\Lambda(w,t) \neq 0$.* This component of $\Lambda(w,t)$ is the part of $C_{13}(w,t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy w and with energy t make a contribution.

The second component $\Phi_1(w,t)$ is the part of $C_{13}(w,t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy w and inelastic scattering with energy t make a contribution, so that we have

$$\Phi_1(w, t) = 0, \quad \text{if} \quad w < 4m^2 \quad \text{or} \quad t < 9\mu^2. \tag{65}$$

We write the unitarity condition now for the reaction (C.III) in the approximation of elastic scattering:†

$$C_{13}^{(3)}(w, t) = -\frac{1}{4\pi^2 t^{1/2} (t/4 - \mu^2)^{1/2}} \int_{(w_1 < w)} dx\, dy \frac{\widetilde{B}_1^*(x, t) \widetilde{B}_1(y, t)}{[(w - w_1)(w - w_2)]^{1/2}},$$

$$C_{23}^{(3)}(\xi, \eta) = C_{13}^{(3)}(\xi, \eta), \tag{66}$$

where

$$w_{1,2} = w_{1,2}(t, x, y) = x + y + \frac{2}{t - 4\mu^2} \{(x - m^2 + \mu^2)(y - m^2 + \mu^2) \pm [(x - m^2 + \mu^2)^2 + x(t - 4\mu^2)]^{1/2} [(y - m^2 + \mu^2)^2 + y(t - 4\mu^2)]^{1/2}\}, \tag{67}$$

and the integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude $C_{13}^{(3)}(w,t)$ consists of two components:

$$C_{13}^{(3)}(w, t) = \Lambda'(w, t) + \Phi_2(w, t). \tag{68}$$

The first component corresponds to the case in which both variables of integration are equal to m^2. It is obvious that

$$\Lambda'(w,t) = -(g^4/4) t^{-1/2} (t/4 - \mu^2)^{-1/2} [w - w_1(t; m^2, m^2)]^{-1/2} \times [w - w_2(t; m^2, m^2)]^{-1/2} \tag{69}$$

for those w and t for which $\Lambda'(w,t) \neq 0$. It is easy to show, therefore, that

$$\Lambda'(w, t) = \Lambda(w, t). \tag{70}$$

The second component in (68) is the part of $C_{13}(w,t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy t and inelastic scattering with energy w make a contribution, such that we have

$$\Phi_2(w, t) = 0, \quad \text{if} \quad w < (2m + \mu)^2 \quad \text{or} \quad t < 4\mu^2. \tag{71}$$

Taking into account Eqs. (54), (63), (66), (68), and (70) we have, consequently,

$$C_{13}(w, t) = \Lambda(w, t) + \Phi_1(w, t) + \Phi_2(w, t) + X_C(w, t), \tag{72}$$

$$C_{23}(\bar{t}, t) = \Lambda(\bar{t}, t) + \Lambda(t, \bar{t}) + \Phi_2(\bar{t}, t) + \Phi_2(t, \bar{t}) + Y_C(\bar{t}, t), \tag{73}$$

where $X_C(w,t)$ is the part of $C_{13}(w,t)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy w and with energy t make a contribution, and $Y_C(\bar{t},t)$ $[Y_C(\bar{t},t) = Y_C(t,\bar{t})]$ is the part of $C_{23}(\bar{t},t)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy $\bar{t}$ and with energy t make a contribution, so that we certainly have

$$X_C(w, t) = 0, \quad \text{if} \quad w < (2m + \mu)^2 \quad \text{or} \quad t < 9\mu^2, \tag{74}$$

$$Y_C(\bar{t}, t) = 0, \quad \text{if} \quad \bar{t} < 9\mu^2 \quad \text{or} \quad t < 9\mu^2. \tag{75}$$

Finally, substituting (72) and (73) in (57), we obtain

$$C_3(t, w) = C_3^e(t, w) + C_3^u(t, w) + C_3^{uu}(t, w), \tag{76}$$

where

$$C_3^e(t, w) = \frac{1}{\pi} \int dx\, [\Phi_2(x, t) + \Lambda(x, t)] \left(\frac{1}{x - w} + \frac{1}{x - \bar{t}}\right), \tag{77}$$

$$C_3^u(t, w) = \frac{1}{\pi} \int dx \left[\frac{\Phi_1(x, t)}{x - w} + \frac{\Phi_2(t, x) + \Lambda(t, x)}{x - \bar{t}}\right], \tag{78}$$

$$C_3^{uu}(t, w) = \frac{1}{\pi} \int dx \left[\frac{X_C(x, t)}{x - w} + \frac{Y_C(x, t)}{x - \bar{t}}\right]. \tag{79}$$

Equations (60), (66) and (76) — (79) are the desired set of Mandelstam equations in the case of scattering of nucleons on nucleons.

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*The boundary of the region where $\Lambda(w, t) = 0$ will be established in Sec. 6, but naturally we have $\Lambda(w, t) = 0$ if $w < 4m^2$ or $t < 4\mu^2$.

†In view of the symmetry of the reactions (C.II) and (C.III), we need not consider the unitarity condition for the reaction (C.II). We shall assume that the unitarity condition for the reaction (C.III) is also valid even in the non-physical energy region $4\mu^2 < t < 4m^2$.

II. METHOD OF SUCCESSIVE EXTENSION OF THE RANGE OF KNOWLEDGE OF SPECTRAL FUNCTIONS

4. Scattering of Pions on Pions

We assume that the partial amplitudes $h_l(\nu^2)$ are given us for the reactions A.I), so that in its physical part

$$A_1(\sigma_1, \sigma_3) = \sum_l \operatorname{Im} h_l(\nu^2)\, P_l(\cos\chi), \tag{80}$$

where l is the angular momentum, and ν^2 and χ are the square of the momentum and the scattering angle for this reaction in the c.m.s., connected with the variables σ_1, σ_2, σ_3 by the relations

$$\sigma_1 = 4\mu^2 + 4\nu^2, \qquad \sigma_2 = -2\nu^2(1 + \cos\chi),$$
$$\sigma_3 = -2\nu^2(1 - \cos\chi). \tag{81}$$

We continue analytically the expression on the right side of (80) in $\cos\chi$ for $\nu^2 = \text{const}$, finding $A_1(\sigma_1, \sigma_3)$ in the entire region where

$$A_{13}(\sigma_1, \sigma_3) = 0, \qquad A_{12}(\sigma_1, \sigma_2) = 0. \tag{82}$$

We shall call this region the region of zero approximation of the absorption part.

It is easy to find where the equations of (82) are satisfied. In view of (4), it is sufficient for us to establish where $A_{13}(\sigma_1, \sigma_3) = 0$ or where $\chi(\sigma_3, \sigma_1) + \chi(\sigma_1, \sigma_3) = 0$. But on the basis of (9), (12) and (13) it can be established that $\chi(\sigma_3, \sigma_1) = 0$ if $\sigma_3 \leq \xi_1(\sigma_1; 4\mu^2, 4\mu^2)$. Therefore, $A_{13}(\sigma_1, \sigma_3) = 0$ in the region where

$$\sigma_3 \leqslant \xi_1(\sigma_1;\ 4\mu^2,\ 4\mu^2), \qquad \sigma_1 \leqslant \xi_1(\sigma_3;\ 4\mu^2,\ 4\mu^2).$$

We shall denote the boundary of this region by $\sigma_1 = \sigma_1^{(0)}(\sigma_3)$ or $\sigma_3 = \sigma_1^{(0)}(\sigma_1)$.

Knowing the absorption part $A_1(\sigma_1, \sigma_3)$ in the region of its zero approximation, we can find $\chi(\sigma_3, \sigma_1)$ by Eq. (12) in the region where $\sigma_3 \leq \xi_1(\sigma_1; \sigma_1^{(0)}(\sigma_1), 4\mu^2)$. Consequently, we can find the spectral function $\chi(\sigma_3, \sigma_1) + \chi(\sigma_1, \sigma_3)$ in the region where

$$\sigma_3 \leqslant \xi_1(\sigma_1;\ \sigma_1^{(0)}(\sigma_1),\ 4\mu^2), \qquad \sigma_1 \leqslant \xi_1(\sigma_3;\ \sigma_1^{(0)}(\sigma_3),\ 4\mu^2).$$

We shall call this region the region of zero approximation of the spectral function $A_{13}(\sigma_1, \sigma_3)$, and shall denote its boundary by $\sigma_1 = \sigma_1^{(1)}(\sigma_3)$ or $\sigma_3 = \sigma_1^{(1)}(\sigma_1)$.

On the basis of (4), we can now find the spectral function $A_{12}(\sigma_1, \sigma_2)$ in the region of its zero approximation. The region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$ of course includes within itself the region of zero approximation of the absorption part $A_1(\sigma_1, \sigma_3)$.

We shall show how we can now find $A_1(\sigma_1, \sigma_3)$ in the entire region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$. The latter we shall therefore call the region of first approximation of the absorption part $A_1(\sigma_1, \sigma_3)$.

With this purpose, we introduce the auxiliary function $A_1'(\sigma_1, \sigma_3)$, which will appear on the right side of Eq. (6) if we set $A_{13}(\sigma_1, \sigma_3) = 0$ and $A_{12}(\sigma_1, \sigma_2) = 0$ outside the regions of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$, respectively. We shall consider the function $A_1'(\sigma_1, \sigma_3)$ to be known. Further, we shall consider the function $A_1(\sigma_1, \sigma_3) - A_1'(\sigma_1, \sigma_3)$. For $\sigma_1 = \text{const}$, this function is already analytic in σ_3 in the region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$. Therefore, it can be obtained there by means of analytic continuation in $\cos\chi$ for $\nu^2 = \text{const}$ from the region of zero approximation of the absorption part $A_1(\sigma_1, \sigma_3)$. It can then be obtained in the region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$, and $A_{12}(\sigma_1, \sigma_2)$ and the absorption part $A_1(\sigma_1, \sigma_3)$.

By extending the region in which the absorption part $A_1(\sigma_1, \sigma_3)$ is known, we can by Eq. (12) extend the region in which the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$ are known which, in turn, makes it possible again to extend the region in which the absorption part $A_1(\sigma_1, \sigma_3)$ is known.

It is clear that the spectral function $A_{13}(\sigma_1, \sigma_3)$ can be found exactly by such a successive extension of the region of its knowledge only when $X_A(\sigma_1, \sigma_3) = 0$.

5. Scattering of Pions on Nucleons

We shall first assume that the partial amplitudes $f_l(k^2)$ are known to us for the reaction (B.I), so that in its physical region

$$B_1(s, u) = \sum_l \operatorname{Im} f_l(k^2)\, P_l(\cos\varphi), \tag{83}$$

where l is the angular momentum, and k^2 and φ are the squares of the momentum and scattering angles for this reaction in the c.m.s., connected with the variables s and u by the relations

$$s = [(k^2 + m^2)^{1/2} + (k^2 + \mu^2)^{1/2}]^2,$$
$$u = -2k^2(1 - \cos\varphi). \tag{84}$$

Second, we shall assume that the analytic continuations of the partial amplitudes $g_l(q^2)$ are given to us for the reaction (B.III) when $q^2 > 0$, so that we have for $q^2 > 0$ and $\cos\vartheta \approx 0$,*

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*It will be established below for what values of $\cos\vartheta$ is (85) valid.

$$B_3(u, s) = \sum_l \operatorname{Im} g_l(q^2)\, P_l(\cos\vartheta), \tag{85}$$

where l is the angular momentum and q^2 and ϑ are the squares of the momentum of the meson and the scattering angle for this reaction in the c.m.s., connected with the variables s and u by the relations

$$s = m^2 - \mu^2 - 2q^2 + 2q\,(q^2 + \mu^2 - m^2)^{1/2}\cos\vartheta,$$

$$u = 4q^2 + 4\mu^2. \tag{86}$$

By continuing the expression on the right side of Eq. (83) analytically in $\cos\varphi$ for $k^2 = \text{const}$, and the expression on the right side of Eq. (85) in $\cos\vartheta$ for $q^2 = \text{const}$, we find $B_1(s, u)$ over the entire region where

$$B_{13}(s, u) = 0, \quad B_{12}(s, \bar{s}) = 0^* \tag{87}$$

and $B_3(u, s)$ over the entire region where

$$B_{13}(s, u) = 0, \quad B_{23}(\bar{s}, u) = 0. \tag{88}$$

We shall call these regions respectively the regions of zero approximation of the absorption part $B_1(s, u)$ and of the absorption part $B_3(u, s)$.

It is easy to find where Eqs. (87) and (88) are satisfied. On the basis of (27), (29) — (35), and (38) — (40) it can be proved that $\psi_1(s, u) = 0$ if $u \le u_1(s; 4\mu^2, 4\mu^2)$ [here, as is easy to show, $u \le u_3(s; m^2, m^2)$] $\psi_2(s, \bar{s}) = 0$ if $\bar{s} \le \bar{s}_1(s; m^2, 4\mu^2)$; $\psi_3(s, u) = 0$ if $s \le s_1(u; m^2, 4\mu^2)$. Therefore, $B_{13}(s, u) = 0$ in the region where

$$u \leqslant u_1(s;\ 4\mu^2,\ 4\mu^2), \qquad s \leqslant s_1(u;\ m^2,\ 4\mu^2),$$

and $B_{12}(s, u) = 0$ in the region where

$$\bar{s} \leqslant \bar{s}_1(s;\ m^2,\ 4\mu^2), \qquad s \leqslant \bar{s}_1(\bar{s};\ m^2,\ 4\mu^2).$$

The boundaries of these regions will be denoted by $s = s^{(0)}(u)$ or $u = u^{(0)}(s)$ and by $\bar{s} = \bar{s}^{(0)}(s)$ or $s = s^{(0)}(\bar{s})$.

Knowing the absorption parts $B_1(s, u)$, $B_3(u, s)$, $B_2(\bar{s}, u)$ [the latter by virtue of (28)] in the region of their zero approximation, and the absorption part $A_1(y, u)$ in the region of its n-th approximation $(n \ge 0)$, we can by Eqs. (29), (30) and (38) find $\psi_1(s, u)$ in the region where $u \le u_1(s; u^{(0)}(s), 4\mu^2)$ [here, as is easy to show, $u \le u_3(s; m^2, m^2)$ all the same]; $\psi_2(s, \bar{s})$ — in the region where $\bar{s} \le \bar{s}_1(s; s^{(0)}(s), 4\mu^2)$ and $\bar{s} \le \bar{s}_1(s; m^2, u^{(0)}(s))$, and $\psi_3(s, u)$ — in the region where $s \le s_1(u; s^{(0)}(u), 4\mu^2)$, and $s \le s_1(u; m^2, \sigma_1^{(n)}(u))$. Consequently, we can find the spectral functions $\psi_1(s, u) + \psi_3(s, u)$ in the region where

$$u \leqslant u_1(s;\ u^{(0)}(s),\ 4\mu^2),\ s \leqslant s_1(u;\ s^{(0)}(u),\ 4\mu^2),$$
$$s \leqslant s_1(u;\ m^2,\ \sigma^{(n)}(u)),$$

and the spectral function $\psi_2(s, \bar{s}) + \psi_2(\bar{s}, s)$ in the region where

$$\bar{s} \leqslant \bar{s}_1(s;\ \bar{s}^{(0)}(s),\ 4\mu^2), \qquad \bar{s} \leqslant \bar{s}_1(s;\ m^2,\ u^{(0)}(s)),$$

$$s \leqslant \bar{s}_1(\bar{s};\ \bar{s}^{(0)}(\bar{s}),\ 4\mu^2), \qquad s \leqslant \bar{s}_1(\bar{s};\ m^2,\ u^{(0)}(\bar{s})).$$

We shall call these regions respectively the regions of zero approximation of the spectral function $B_{13}(s, u)$ and the spectral function $B_{12}(s, \bar{s})$, and the boundaries of them we shall denote by $s = s^{(1)}(u)$ or $u = u^{(1)}(s)$ and by $\bar{s} = \bar{s}^{(1)}(s)$ or $s = s^{(1)}(\bar{s})$.

On the basis of (23), we can now find the spectral function $B_{23}(\bar{s}, u)$ in the region of its zero approximation. The region of zero approximation of the spectral functions $B_{13}(s, u)$ and $B_{12}(s, \bar{s})$ naturally contain within themselves the region of the zero approximation of the absorption part $B_1(s, u)$, while the region of zero approximation of the spectral functions $B_{13}(s, u)$ and $B_{23}(\bar{s}, u)$ contain within themselves the region of zero approximation of the absorption part $B_3(u, s)$.

Continuing in the same fashion as in the case of the scattering of π mesons on π mesons, we can now find $B_1(s, u)$ throughout the entire region of zero approximation of the spectral functions $B_{13}(s, u)$ and $B_{12}(s, \bar{s})$, and also $B_3(u, s)$ throughout the entire region of the zero approximation of the spectral functions $B_{13}(s, u)$ and $B_{23}(\bar{s}, u)$. We shall therefore call these latter the regions of first approximation of the absorption part $B_1(s, u)$ and the absorption part $B_3(u, s)$, respectively.

Extending the region in which the absorption parts $B_1(s, u)$ and $B_3(u, s)$ are known, we can, by Eqs. (29), (30), and (38), extend the region in which the spectral functions $B_{13}(s, u)$, $B_{12}(s, \bar{s})$ and $B_{23}(\bar{s}, u)$ are known; in turn this makes it possible again to extend the region in which the absorption parts $B_1(s, u)$ and $B_3(u, s)$ are known.

It is clear that the spectral functions $B_{13}(s, u)$ and $B_{12}(s, \bar{s})$ can be found exactly in such a stepwise extension of the region of their knowledge only when $X_B(s, u) = 0$ and $Y_B(s, \bar{s}) = 0$.

6. Scattering of Nucleons on Nucleons

Making use of the unitarity condition for the reaction (C.III), we shall have for $4\mu^2 < t < 9\mu^2$* and $\cos\psi \approx 0$:†

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*And only in the part of this region which is symmetric in $\cos\varphi$, if $B_1(s, u)$ is analytically continued in the form of an expansion in Legendre polynomials.

*We again assume the validity of the unitarity condition for the reaction (C.III) in the non-physical region.

†It will be established below for what values of $\cos\psi$ is (89) valid.

$$C_3(t, w) = C_3^e(t, w) = \frac{q}{(q^2+\mu^2)^{1/2}} \sum_l |g_l(q^2)|^2 P_l(\cos\psi), \quad (89)$$

where q^2 and ψ are the squares of the momentum of the meson and the angle of scattering for this reaction in the c.m.s., which are connected with the variables w and t by the relations

$$w = -2(q^2+\mu^2-m^2)(1+\cos\psi), \quad t = 4q^2+4\mu^2. \quad (90)$$

Analytically continuing the expression on the right hand side of Eq. (89) in $\cos\psi$ for $q^2 = \text{const}$, we find $C_3(t, w) = C_3^e(t, w)$ over the entire region where $4\mu^2 < t < 9\mu^2$ and where

$$C_{13}(w, t) = 0, \quad C_{23}(\bar{t}, t) = 0. \quad (91)$$

We shall call this region the region of zero approximation for the absorption part $C_3(t, w)$.

It is easy to establish where the equations of (91) are satisfied. On the basis of (58) and (60) — (63) it can be established that $\Lambda(w, t) = 0$ if $t \le t_1(w; \mu^2, \mu^2)$. Therefore, $C_{13}(w, t) = 0$ in the region where $t \le t_1(w; \mu^2, \mu^2)$, while $C_{23}(\bar{t}, t) = 0$ in the region in which $t \le t_1(\bar{t}; \mu^2, \mu^2)$ and $\bar{t} \le t_1(t; \mu^2, \mu^2)$.

Knowing the absorption parts $B_1(s, t)$ in the region of its n-th approximation ($n \ge 0$), we can find $\Phi_2(w, t)$ from the first of Eqs. (66) in the region in which $w \le w_1(t; s^{(n)}(t), m^2)$. The functions $\Phi_2(w, t)$ can thus be assumed to be given in the case of scattering of nucleons on nucleons. But then we can find the spectral function $C_{13}(w, t)$ in the region where $4\mu^2 < t < 9\mu^2$ and $w \le w_1(t; s^{(n)}(t), m^2)$, and the spectral function $C_{23}(\bar{t}, t)$ in the region where $4\mu^2 < t < 9\mu^2$ and $\bar{t} \le w_1(t; s^{(n)}(t), m^2)$. These regions, which are identical for sufficiently high n with the region $4\mu^2 < t < 9\mu^2$, we shall call the regions of zero approximation of the spectral function $C_{13}(w, t)$ and the spectral function $C_{23}(\bar{t}, t)$, respectively. These regions of course include the region of zero approximation of the absorption part $C_3(t, w)$.

Proceeding as above, we can now find $C_3(t, w)$ throughout the entire region of the zero approximation of the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$. We shall also call the latter the region of first approximation of the absorption part $C_3(t, w)$.

Knowing the absorption part $C_3(t, w)$ for sufficiently large n in the region of its first approximation, we can, by Eq. (60), find $\Phi_1(w, t)$ throughout the entire region where $9\mu^2 < t < 16\mu^2$. We shall call this region the region of first approximation of the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$.

Making use of Eqs. (77) and (78), we can now find $C_3^e(t, w)$ and $C_3^u(t, w)$ in the region of first approximation of spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$. We shall also call the latter the region of second approximation of the absorption part $C_3(t, w)$.

Extending the region in which the absorption part $C_3(t, w)$ is known, we can by Eq. (60) extend the region in which the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$ are known; in turn, this again enables us to extend the region in which the absorption part $C_3(t, w)$ is known.

It is clear that the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$ can be found exactly in such a stepwise extension of the region of their knowledge only when $X_C(w, t) = 0$ and $Y_C(\bar{t}, t) = 0$.

7. Discussion

By slightly altering the method set forth in Sec. 4, we can proceed to construct integral equations for the amplitudes of the reactions A, B, and C in their physical regions.

Assuming also that the absorption part $C_1(w, t)$ is known in the physical region of the reaction (C.I), and proceeding just as in the case of ππ and πN scattering, we could have successively found the values for the absorption parts of the reactions A, B, and C over wider and wider regions by means of a knowledge of them in the physical regions. In this case, we could have completely discarded the unknown parts of the spectral functions X and Y, restricting ourselves to the problem of determining that part of the amplitude which does not contain graphs which are inelastic in both channels. Then expressing the amplitude in terms of the absorption part, we could have obtained the integral equations for the amplitudes in the physical regions.

Such equations require simplification for practical use. In particular, rewriting the equations of Mandelstam for a fixed angle of scattering for any reaction, and then transforming to the partial amplitudes, we can easily obtain an equation similar to that given by Cini and Fubini.[5] The series of additional terms appearing in this case both on the right and the left of the cut make the approximation of Cini and Fubini doubtful. The obtaining of more convincing estimates appears possible to us in our approach to the construction of integral equations; these estimates are connected with the neglect of isolated components. These problems, however, go beyond the framework of the present paper.

Keeping in mind the calculation which is now under way of a real case in which spin and charge variables are present, we shall make a concluding remark relative to the capability of inclusion of the phases of nucleon-antinucleon annihilation into two π mesons. It appears most advantageous to us to connect them with the phases of $\pi\pi$ and πN scattering by means of a solution by the method of Muskhelishvili[6] of an integral equation for the amplitude of πN scattering for a fixed angle of scattering of the reaction (B.III). Thus one can choose the phases of the nucleon-antinucleon annihilation into two π mesons from the phase of $\pi\pi$ scattering and from the absorption part $B_1(s, u)$. The latter can be expressed through the pole term* and through the phase shifts of $\pi\pi$ and πN scattering.

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*Only this pole term was considered by Mandelstam.[4]

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ON THE ASYMPTOTIC BEHAVIOR OF GREEN'S FUNCTIONS IN QUANTUM FIELD THEORY

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The formalism of vacuum expectation values of products of field operators[4-6] is used to prove for quantum field theory a principle of the weakening of correlations analogous to the corresponding Bogolyubov principle in quantum statistics. By the extension of these results to the Green's functions, a proof is obtained for the hypothesis of Freese about the asymptotic behavior of Green's functions at large space separations.

THE study of the properties of Green's functions in quantum field theory is an extremely difficult problem. The question arises of approximating them by Green's functions of the lowest orders, whose properties have been most completely studied. This question has been treated in papers by Freese[1] and Falk.[2] Their conclusions are based on the hypothesis that one can neglect the interaction at points separated by an infinite distance.

The present paper is devoted to a rigorous proof and a refinement of the hypothesis of Freese about the behavior of Green's functions. In our deductions we do not make the assumption that interactions at large distances are small. In the statement of the problem we start from an idea of Bogolyubov.[3] In his lectures on statistical physics Bogolyubov has recently formulated a principle of the weakening of the correlations between particles for systems in the state of statistical equilibrium. In our paper this principle is formulated for quantum field theory and we give a proof of it on the basis of the axioms of the theory.

Our conclusions are based on the study of the behavior of vacuum expectation values of products of operators. For this purpose we use the apparatus of functions developed by Wightman, Källen, and Wilhelmsson.[4-6] The behavior of vacuum expectation values for fixed equal time components has been considered by Dell'Antonio and Gulmanelli[7] in a study of the spatial asymptotic condition of Haag.[8,9]

1. THE PRINCIPLE OF WEAKENING OF CORRELATIONS

Let us consider a neutral scalar field which is described by a Hermitian operator $A(x)$. By hypothesis the operator $A(x)$ satisfies the conditions of causality and of translational and Lorentz invariance. It is also assumed that there exists a unique normalized vacuum state $|0\rangle$ and that there are no states with negative energies.

Let us consider a set of points M consisting of $n+1$ vectors x_i, where $x_i = (\mathbf{x}_i, x_i^0)$ and the scalar product is defined in the following way:

$$x_i^2 = (x_i^1)^2 + (x_i^2)^2 + (x_i^3)^2 - (x_i^0)^2 \, .$$

Let us divide the set of points M into two subsets M_1 and M_2:

$$M = M_1 + M_2, \quad \mathbf{x}_\alpha \in M_1, \quad \mathbf{x}_\beta \in M_2;$$

$$\alpha = 1, 2, \ldots, i; \quad \beta = i+1, \ldots, n+1.$$

Let the sets M_1 and M_2 be such that each of them taken separately can be inclosed in some sphere of finite radius. Let us now make these spheres undergo a displacement relative to each other. In doing so we shall assume that the configurations of the points of the sets M_1 and M_2 change only inside the respective spheres, and that the time components of the vectors x_i are arbitrarily fixed. The analog of Bogolyubov's principle of the weakening of correlations[3] holds for this case.

The vacuum average

$$f(x_1, \ldots, x_{n+1}) = \langle 0 | A(x_1) \ldots A(x_i) A(x_{i+1}) \ldots A(x_{n+1}) | 0 \rangle \qquad (1)$$

separates into a product of vacuum averages

$$\langle 0 | A(x_1) \ldots A(x_i) | 0 \rangle \langle 0 | A(x_{i+1}) \ldots A(x_{n+1}) | 0 \rangle, \qquad (2)$$

if the set of points M_1 goes to an infinite distance from the set M_2 in the sense defined above.

For the proof we make use of the results of

Källen, Wightman, and Wilhelmsson.[4-6] We expand the vacuum average (1) in terms of a complete system of intermediate states

$$f(x_1, \ldots, x_{n+1}) = \sum_{|z_k>} \exp\{i\Sigma p^{(z_k)}\xi_k\}$$

$$\times \langle 0 | A_1 | z_1\rangle \langle z_1 | A_2 | z_2\rangle \ldots \langle z_n | A_{n+1} | 0\rangle, \tag{3}$$

where $p^{(z_k)}$ is the eigenvalue of the operator of energy-momentum of the state $|z_k>$, and moreover

$$\xi_k = x_k - x_{k+1}, \langle z_{k-1} | A_k(x_k) | z_k\rangle$$

$$= \langle z_{k-1} | A_k | z_k\rangle \exp\{i(p^{(z_k)} - p^{(z_{k-1})})x_k\}. \tag{4}$$

Let us rewrite the relation (3) in the form

$$f(x_1, \ldots, x_{n+1}) = \langle 0 | A(x_1) \ldots A(x_i) | 0\rangle \langle 0 |$$

$$\times A(x_{i+1}) \ldots A(x_{n+1}) | 0\rangle + f_1, \tag{5}$$

where

$$f_1 = \sum_{|z_1>\ldots|z_i>\ldots|z_n>} \exp\left\{i\sum p^{(z_k)}\xi_k\right\} \langle 0 | A_1 | z_1\rangle$$

$$\times \langle z_1 | A_2 | z_2\rangle \ldots \langle z_n | A_{n+1} | 0\rangle, \tag{6}$$

with the sum taken over all intermediate states with the exception of the states $|z_i>$, the sum over which is taken starting with the one-particle state.

It can be seen from the relation (5) that for our purpose it is enough to prove the equation

$$\lim_{R\to\infty} R^m f_1 = 0, \tag{7}$$

where R is the distance between the spheres that were introduced above and m is an arbitrary positive integer.

Starting from the conditions of our problem, we shall find out certain properties of the function f_1. If we denote the Fourier transform of the function f_1 by $G(p_1, \ldots, p_n)$, then we can write

$$f_1 = \frac{1}{(2\pi)^{3n}} \int \cdots \int dp_1 \ldots dp_n$$

$$\times \exp\left\{i\sum_{k=1}^{n} p_k\xi_k\right\} G(p_1, \ldots, p_n). \tag{8}$$

In virtue of invariance under Lorentz transformations the function $G(p_1, \ldots, p_n)$ is a function of all possible scalar products of the vectors $p_1, \ldots, p_n$. In virtue of the condition that the energy is positive the function $G(p_1, \ldots, p_n)$ is different from zero only when all of the vectors p_k lie in the upper light cone, and furthermore the vector p_i must lie in the upper hyperboloid $p_i^2 \leq -\alpha_{ii}$, $(p_i^0)^2 \geq \alpha_{ii} > 0$, which is also in the corresponding upper light cone. Here α_{ii} is defined by the relation $p_i^2 = \alpha_{ii}$, under the condition that the vector p_i corresponds to a one-particle state.

The fact that the vector p_i lies in the upper hyperboloid is due to the sum in Eq. (6) being taken over states $|z_1>$ beginning with the one-particle state. What we have said can be written in the following way:

$$G(p_1, \ldots, p_n)$$

$$= G(p_1^2, \ldots, p_kp_l, \ldots, p_n^2) \prod_{k=1,\, k\neq i}^{n} \theta(p_k)\,\theta(p_i^0 - \alpha_{ii});$$

$$\theta(p_k) = \frac{1}{2}\left[1 + \frac{p_k^0}{|p_k^0|}\right].$$

We can now write the relation (8) in the form

$$f_1 = i^n \int_{\alpha_{kl}}^{\infty} \cdots \int \prod_{k,\,l=1}^{n} da_{kl} G(-a_{11}, \ldots, -a_{kl}, \ldots, -a_{nn})$$

$$\times \Delta_{n+1}^{(+)}(\xi_k; a_{kl}), \tag{9}$$

where

$$\Delta_{n+1}^{(+)}(\xi_k; a_{kl}) = \frac{(-i)^n}{(2\pi)^{3n}} \int \cdots \int dp_1 \ldots dp_n$$

$$\times \exp\left\{i\sum_{k=1}^{n} p_k\xi_k\right\} \prod_{k\leqslant l=1}^{n} \delta(p_kp_l + a_{kl}) \prod_{k=1}^{n} \theta(p_k). \tag{10}$$

The quantities α_{kl} are zero, except that $\alpha_{ii} > 0$.

It can be seen from Eq. (9) that for the proof of the equation (7) it is enough to establish the relation

$$\lim_{R\to\infty} R^m \Delta_{n+1}^{(+)}(\xi_k; a_{kl}) = 0. \tag{11}$$

In this connection one also has to justify the passage to the limit under the signs of integration in Eq. (9).

Thus the proof of the principle of the weakening of correlations has been reduced to the study of the asymptotic behavior of the singular functions of Källen and Wilhelmsson.[6] We can study the asymptotic behavior of these functions only in the region where they exist. Therefore it is also necessary to show that the passage to the limit R → ∞ occurs in the region of existence.

2. PROOF OF THE RELATION (11)

Källen and Wilhelmsson[6] have obtained the following representation for the function $\Delta_{n+1}^{(+)}(\xi; a_{kl})$:

$$\Delta_{n+1}^{(+)}(\xi_k; a_{kl}) = \frac{(-i)^{n-4}}{(2\pi)^{3(n-4)}} (-D)^{(n-4)^2/2} \prod_{k_1\leqslant k_2=5}^{n} \delta(Da_{k_1k_2}$$

$$- \sum_{\lambda,\,\lambda'=1}^{4} a_{k_1\lambda}a_{\lambda\lambda'}a_{\lambda'k_2})\, \Delta_5^{(+)}(y_\varkappa; a_{\varkappa\varkappa'}) \prod_{k=5}^{n} \theta(a_{1k}),$$

$$y_\varkappa = x_\varkappa + \sum_{\lambda=1}^{4}\sum_{k=5}^{n} D^{-1}\Delta_{\varkappa\lambda}a_{\lambda k}x_k,$$

$$D = \operatorname{Det}\|a_{ik}\| < 0 \quad (1 \leqslant i, k \leqslant 4),$$

$$D_0 = \operatorname{Det}\|a_{ik}\| \quad (1 \leqslant i, k \leqslant 3). \tag{12}$$

The function $\Delta_5^{(+)}$ has the form

$$\Delta_5^{(+)}\ (\xi\ ;\ a_{kl}) = \frac{1}{(2\pi)^{12}}\frac{1}{\sqrt{-D}}\theta$$

$$\times\ (-D)\ \theta\ (D_0)\ \theta\ (a_{12}^2 - a_{11}a_{22})\ \prod_{k=2}^{4}\theta\ (a_{1k})\ I; \tag{13}$$

$$I = I^{(1)} + I^{(2)}, \qquad I^{(1,2)} = \frac{(2\pi)^3}{2}\int_{-\infty}^{\infty}\frac{t\,dt H_0^{(1)}(t)}{\sqrt{\Sigma_{1,2}(t)}}; \tag{14}$$

$$\Sigma_{1,2}(t) = [(t^2 - Q)^2 - P \pm \sqrt{T}]^2 - t^2S \mp 8\sqrt{T}t^2\,(t^2 - Q),$$

$$Q = I_1, \qquad P = 2\,(I_1^2 - I_2), \qquad S = \tfrac{32}{3}(I_1^3 - 3I_1I_2 + 2I_3),$$

$$T = \tfrac{8}{3}(I_1^4 - 6I_1^2I_2 + 3I_2^2 + 8I_3I_1 - 6I_4),$$

$$I_k = \mathrm{Sp}\ [(AX)^k]; \qquad k = 1, 2, 3, 4. \tag{15}$$

Here A and X denote the matrices $\| a_{kl} \|$ and $\| \xi_k \cdot \xi_l \|$.

In the passage to the limit $R \to \infty$ the distance between points that belong to the same set, M_1 or M_2, remains less than some constant, which is fixed by the radius of the corresponding sphere. The distances that increase will be those between pairs of points, one of which belongs to the set M_1 and the other to M_2. The function $\Delta_{n+1}^{(+)}(\xi_k;\ a_{kl})$ involves only one such pair of points: $\xi_i = x_i - x_{i+1}$. In the passage to the limit the difference between the coordinates of the points increases in proportion to the quantity R. Consequently, we can introduce a new quantity ξ_i', by setting $\xi_i = (R\xi_i',\ \xi_i^0)$. By the theorem of Hall and Wightman[10] the function $\Delta_{n+1}^{(+)}(\xi_k;\ a_{kl})$, and consequently also the quantities Q, P, S, T, are functions of the scalar products $\xi_k \cdot \xi_l$.

For the further argument it is convenient to introduce the quantities

$$q = Q/R^2, \quad p = P/R^4, \quad s = S/R^6, \quad \tau = T/R^8, \quad t = Rt' \tag{16}$$

The relations (14) and (15) then take the forms

$$I^{(1,2)} = \frac{(2\pi)^3}{2}\frac{1}{R^2}\int_{-\infty}^{\infty}\frac{t'\,dt' H_0^{(1)}(Rt')}{\sqrt{\Sigma_{1,2}(t')}}, \tag{17}$$

$$\Sigma_{1,2}(t') = [(t'^2 - q)^2 - p \pm \sqrt{\tau}]^2 - t'^2s \mp 8\sqrt{\tau}t'^2\,(t'^2 - q). \tag{18}$$

Direct calculation of the integrals (17) is extremely complicated.

Let us first find out the properties of the functions q, p, s, τ, starting from the conditions of our problem. When expressed in terms of the matrix elements $(AX)_{ij}$ the functions I_k have the form

$$I_k\,(AX)_{ii}^k + R_k, \quad k = 1, 2, 3, 4, \tag{19}$$

where R_k contains all the other terms, and

$$(AX)_{ii} = a_{1i}\xi_i \cdot \xi_1 + a_{2i}\,\xi_i \cdot \xi_2 + \ldots + a_{ii}\xi_i^2$$
$$+ \ldots + a_{ni}\xi_i \cdot \xi_n. \tag{20}$$

For the further argument it is convenient to write this relation in the form

$$(AX)_{ii} = a_{ii}\xi_i^2 + r_1, \tag{21}$$

where the function r_1 contains all the remaining terms.

Calculating the functions Q, P, S, T by means of Eq. (19) and subsituting the expressions for these functions in Eq. (16), we find

$$q = a_{ii}\xi_i^2 + R_1'/R, \tag{22}$$

$$p = R_2'/R, \qquad s = R_3/R, \qquad \tau = R_4'/R. \tag{23}$$

It follows from the relations (23) that in the limit $R \to \infty$ the functions p, s, τ go to zero. According to Källen and Wilhelmsson[6] in this case the function $\Delta_{n+1}^{(+)}$ reduces to the function

$$\Delta_2^{(+)}\,(\xi_i;\ a_{ii}) = -\frac{a_{ii}}{8\pi}H_1^{(1)}(\sqrt{a_{ii}z})/\sqrt{a_{ii}z}, \quad z = -\xi_i^2. \tag{24}$$

This fact can be interpreted in the following way. In the limiting case $R \to \infty$ the correlation between the sets M_1 and M_2 can be interpreted as a correlation between the two points x_i and x_{i+1}. The correlation between them will then naturally be described by the correlation function $\Delta_2^{(+)} \equiv \Delta_2^{(+)}(\xi_i;\ a_{ii})$, which depends only on the differences of the coordinates of these two points. The function $\Delta_2^{(+)}$ is the boundary value of a certain analytic function which is regular in the entire complex plane of the variable z with the exception of a cut along the positive real axis. By the conditions of our problem $a_{ii} > 0$, and for sufficiently large R the vector ξ_i is spacelike. This means that we are studying our function on the negative real axis, which is in the region of existence of the function $\Delta_2^{(+)}$. This gives us the right to calculate this function at the point $R = \infty$.

Taking into account the behavior of the Hankel function $H_1^{(1)}$ in the upper half-plane, we find the relation

$$\lim_{R\to\infty} R^m\Delta_2^{(+)}\,(\xi_i;\ a_{ii}) = 0.$$

Thus we have proved the relation (11).

It is interesting to extend our arguments to other cases, for example to the case in which the set M is divided into three or four subsets. In these cases the study of the function $\Delta_{n+1}^{(+)}$ reduces to the study of the functions $\Delta_3^{(+)}$ and $\Delta_4^{(+)}$, and the proof can be carried through in analogy with the arguments of Dell'Antonio and Gulmanelli.[7]

The case in which the set M is divided into five or more subsets requires special treatment.

To justify the passage to the limit under the integral signs in Eq. (8) one must show that the passage to the limit in Eq. (9) is uniform in all the a_{kl}. It is obvious that we can always take $(\xi_i')^2$ and R so large that all of the arguments carried out above will hold for all a_{kl}.

With this we have completely proved the relation (2).

3. THE ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS

Let us consider the function

$$\langle 0 | T(x_1, \ldots, x_{n+1}) | 0 \rangle = \Sigma \Theta(x_1 - x_2) \ldots \Theta(x_n - x_{n+1}) \times \langle 0 | A(x_1) \ldots A(x_{n+1}) | 0 \rangle, \tag{25}$$

where the sum is taken over all possible permutations of the indices 1, 2, . . . , n + 1. Let us investigate the behavior of this function when the distance between the spheres introduced earlier becomes infinite. We regard the time components as arbitrarily fixed. For our purpose let us apply the principle of the weakening of correlations to each term in the right member of Eq. (25). It is not hard to see that all of the terms take the form (2), because for sufficiently large distance between the spheres the vectors of the sets M_1 and M_2 become spacelike, and consequently in virtue of the condition of local commutativity the operators $A(x_1), \ldots, A(x_i)$ will commute with the operators $A(x_{i+1}), \ldots, A(x_{n+1})$. In the limit the function (25) takes the form

$$\langle 0 | T(x_1, x_2, \ldots, x_i) | 0 \rangle \langle 0 | T(x_{i+1}, \ldots, x_{n+1}) 0 \rangle. \tag{26}$$

With this, we have proved the hypothesis of Freese[1] about the asymptotic behavior of the function (25).

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ON THE THIRRING MODEL

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The four-fermion Thirring model is considered and the field operators are determined by functional quadrature.

1. INTRODUCTION

IN problems connected with second quantization, it is always necessary to consider operators written in normal form

$$\hat{A} = \sum\int K(\xi_1, \ldots, \xi_n \,|\, \eta_1, \ldots, \eta_p)$$
$$\times c^*(\xi_1) \ldots c^*(\xi_n)\, c(\eta_1) \ldots, c(\eta_p)\, d\xi_1 \ldots d\eta_p, \quad (1.1)$$

where $c(\xi)$ and $c^*(\xi)$ are operators that satisfy the canonical (Bose or Fermi) commutation relations. It is natural to set in correspondence with each operator written in the form (1.1) a functional

$$A(f^*, f) = \sum\int K(\xi_1 \ldots \xi_n \,|\, \eta_1 \ldots \eta_p)\, f^*(\xi_1) \ldots f^*(\xi_n)\, f(\eta_1)$$
$$\ldots f(\eta_p)\, d\xi_1 \ldots d\eta_p; \quad (1.2)$$

The functions $K(\xi_1 \ldots \xi_n | \eta_1 \ldots \eta_p)$ are the same as in (1.1). In the Bose case the functions $f(\xi)$, $f^*(\xi)$ are complex (not necessarily complex conjugate), while in the Fermi case f and f^* are functions with anticommuting values

$$\{f(\xi), f(\xi')\} = \{f(\xi), f^*(\xi')\} = \{(f^*(\xi), f^*(\xi')\} = 0.$$

It is obvious that the operator (1.1) is uniquely obtained from the functional (1.2).

Let the operators c and c^* be expressed linearly in terms of the operators a and a^*, which satisfy the same commutation relations as c and c^*:

$$c(\xi) = \int\Phi(\xi, \eta)\, a(\eta)\, d\eta + \int\Psi(\xi, \eta)\, a^*(\eta)\, d\eta,$$
$$c^*(\xi) = \int\overline{\Psi}(\xi, \eta)\, a(\eta)\, d\eta + \int\overline{\Phi}(\xi, \eta)\, a^*(\eta)\, d\eta \quad (1.3)$$

($\overline{\Phi}$ and $\overline{\Psi}$ are the complex conjugates of Φ and Ψ). Substituting c and c^* from (1.3) into (1.1), we arrange a and a^* in the normal sequence. We set the operator $\hat{A}$, written in normal sequence with respect to a and a^*, in correspondence with a functional $\widetilde{A}(\alpha^*, \alpha)$ by the same rule as before. Thus, we find that the canonical transformation (1.3) generates a linear transformation in the space of the functionals. This linear transformation can be written with the aid of the continued integral*

$$\widetilde{A}(\alpha^*, \alpha) = \int\mathscr{K}(\alpha^*, \alpha \,|\, f^*, f)\, A(f^*, f) \prod_{\xi} df^*(\xi)\, df(\xi). \quad (1.4)$$

The kernel $\mathscr{K}$ of this transformation has been calculated earlier[1] for the case of both Bose and Fermi commutation relations. For the case of Bose commutation relations this kernel has the form

$$\mathscr{K}(\alpha^*, \alpha \,|\, f^*, f) = \frac{1}{\sqrt{\det\Psi\Psi^*}} \exp\Big\{-\frac{1}{2}[(\overline{\Phi}\Psi^{-1}b, b)$$
$$+ (\Phi\Psi^{-1}\, b^*, b^*) - 2(b^*b)]\Big\} \quad (1.5)$$

and for the case of Fermi commutation relations

$$\mathscr{K}(\alpha^*, \alpha \,|\, f^*, f) = \sqrt{\det\Psi\Psi^*} \exp\Big\{\frac{1}{2}[(\overline{\Phi}\Psi^{-1}b, b)$$
$$- (\Phi\overline{\Psi}^{-1}b^*, b^*) + 2(b^*b)]\Big\}, \quad (1.6)$$

where Φ, Ψ, $\overline{\Phi}$, and $\overline{\Psi}$ are operators specified respectively by the kernels $\Phi(\xi, \eta)$, $\Psi(\xi, \eta)$, $\overline{\Phi}(\xi, \eta)$, and $\overline{\Psi}(\xi, \eta)$;

$$b = \Phi\alpha + \Psi\alpha^* - f, \qquad b^* = \overline{\Psi}\alpha + \overline{\Phi}\alpha^* - f^*,$$
$$(b_1, b_2) = \int b_1(x)\, b_2(x)\, dx.$$

The matrices of the quadratic forms, contained in the exponent of (1.5) and (1.6), are symmetrical and skew-symmetrical, respectively.[1]

In the present paper we employ formula (1.4) for the Thirring model.[2,3]

2. CANONICAL TRANSFORMATIONS OF THE FUNCTIONALS

1. Before we deal with the Thirring model, let us solve the following general problem. Let $\hat{A}$ be an operator, the normal form of which with respect to the operators c and c^* corresponds to the functional

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*For the case of Fermi commutation relations, the integral (1.4) is defined in the paper of the author[1] or of Khalatnikov.[4]

$$A(f^*, f) = \exp\left\{\frac{1}{2}\left[(A_1 f, f) + (A_2 f^*, f^*) + 2(A_3 f^*, f)\right]\right\}. \tag{2.1}$$

Let us carry out the canonical transformation (1.3) and find the functional corresponding to the operator $\hat{A}$, written in normal form with respect to a and a*. Since only the case of Fermi commutation relations is of importance for the Thirring model, we shall consider this case in detail. The case of Bose commutation relations will not be discussed, and only the answer will be given at the end of this section, to complete the exposition.

2. To carry out the canonical transformation we must calculate the integral (see (1.4) and (1.6)):

$$\widetilde{A}(\alpha^*, \alpha) = (\det \Psi\Psi^*)^{1/2} \int^F \exp \frac{1}{2}\left[(\overline{\Phi}\Psi^{-1}b, b) - (\Phi\overline{\Psi}^{-1}b^*, b^*) + 2(b^*, b) + A_1(f, f) + A_2(f^*, f^*) + 2A_3 f^* f\right] \Pi\, df^* df. \tag{2.2}$$

$\int$ denotes the integral over the anticommuting variables. Its definition will not be given here (see references 1 or 4). In the present investigation the only important fact is that

$$\int^F \exp\left(\frac{1}{2} a_{ik} x^i x^k\right) dx = (\det \| a_{ik} \|)^{1/2}, \quad a_{ik} = -a_{ki}.$$

To shorten the notation we introduce the following symbols:

$$\mathcal{A} = \begin{pmatrix} A_1 & A_3 \\ A_3' & A_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\Phi\overline{\Psi}^{-1} \end{pmatrix},$$

$$\xi = -\begin{pmatrix} f \\ f^* \end{pmatrix}, \quad \eta = \begin{pmatrix} \Phi & \Psi \\ \overline{\Phi} & \overline{\Psi} \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}. \tag{2.3}$$

In this notation, the integrand assumes the form exp (F/2), where

$$F = (\mathcal{B}(\xi + \eta), (\xi + \eta)) + (\mathcal{A}\xi, \xi).$$

It is convenient to write F with the aid of matrix multiplication in the form

$$F = \xi' \mathcal{A} \xi + (\xi + \eta)' \mathcal{B} (\xi + \eta),$$

where the prime denotes the transpose of the matrix. This notation is analogous to that of the case with finite number of dimensions. For finite dimensions $\xi' C \xi = \xi_i C_{ik} \xi_k$. In the general case

$$\xi' C \xi = \int \xi(x)\, \boldsymbol{C}(x, y)\, \xi(y)\, dx\, dy.$$

The following identity can be readily verified

$$\xi' \mathcal{A} \xi + (\xi + \eta)' \mathcal{B} (\xi + \eta) = (x + \xi)' (\mathcal{A} + \mathcal{B})(x + \xi) + \eta' (\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1} \mathcal{B})\eta, \tag{2.4}$$

where $x = (\mathcal{A} + \mathcal{B})^{-1} \mathcal{B}\eta$. Using (2.4) we readily calculate the integral (2.2), and find it to be

$$\widetilde{A}(\alpha^*, \alpha) = (\det \Psi\Psi^*)^{1/2} \left[\det(\mathcal{A} + \mathcal{B})^{1/2} \exp\left[\frac{1}{2}\eta' (\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B})\, \eta\right]\right. . \tag{2.5}$$

We transform this expression in the following manner:

$$\mathcal{A} + \mathcal{B} = \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix} + \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\Phi\overline{\Psi}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\Phi}\Psi^{-1} + A_1 & A_3 - E \\ E - A_3' & A_2 - \Phi\overline{\Psi}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix} \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\overline{\Phi}\Psi^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix} \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix}. \tag{2.6}$$

Hence

$$\mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix} \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1} \times \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix} \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix},$$

$$\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix} \left[E - \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1} \times \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\right] \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}$$

$$\times \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1}$$

$$\times \left[\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix} - \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\right] \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix}.$$

The remainder of the derivation is obvious. We obtain ultimately

$$\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & \overline{\Psi} \\ \Psi & \Phi \end{pmatrix} \begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}^{-1} \times \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix}. \tag{2.7}$$

From (2.6) we find that

$$\det(\mathcal{A} + \mathcal{B}) = (\det \overline{\Psi}\Psi^*)^{-1} \det \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.5), we obtain an expression for the transformed functional

$$\widetilde{A}(\alpha^*, \alpha) = \det \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{1/2} \times \exp\left\{\frac{1}{2}\eta' \begin{pmatrix} \overline{\Phi} & \overline{\Psi} \\ \Psi & \Phi \end{pmatrix} \begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}^{-1} \times \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix} \eta\right\}. \tag{2.9}$$

We note finally that

$$\det \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix} = \det \begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}. \tag{2.10}$$

Actually, it is clear beforehand that the left half is equal to the right one, apart from the sign. In order to determine the sign, let us consider the finite-dimensional approximation of the operators

Φ, Ψ, and A_i. In the finite-dimensional approximation these operators should be given by matrices of even order. In fact, in real Fermi field, there is present along with each particle also an antiparticle, and therefore the sector c_k, which is the finite-dimensional approximation of $c(\xi)$, should have 2n components: n components pertain to the particles and n to the antiparticles. In the case of even-dimensional matrices Φ, Ψ, and A_i, the equality (2.10) is obvious. Thus,

$$\widetilde{A}(\alpha^*,\ \alpha) = (\det C')^{1/2}\exp\left\{\frac{1}{2}\eta'\begin{pmatrix}\Phi & \overline{\Psi}\\ \Psi & \Phi\end{pmatrix}C'^{-1}\begin{pmatrix}A_1 & A_3\\ -A_3' & A_2\end{pmatrix}\eta\right\},$$

$$C' = \begin{pmatrix}\overline{\Phi}+A_1\Psi & (E-A_3)\overline{\Psi}\\ (E-A_3')\Psi & \Phi-A_2\overline{\Psi}\end{pmatrix}. \qquad (2.9')$$

The matrix in the exponent of (2.9′) is skew-symmetrical, since it is equal to (½) $[\mathcal{B}-\mathcal{B}(\mathcal{A}+\mathcal{B})^{-1}\mathcal{B}]$, where $\mathcal{A}$ and $\mathcal{B}$ are skew-symmetrical matrices. Using this fact, we can write $\widetilde{A}(\alpha^*,\alpha)$ in the form*

$$\widetilde{A}(\alpha^*,\ \alpha) = (\det C)^{1/2}\exp\left\{\frac{1}{2}\eta'\begin{pmatrix}A_1 & A_3'\\ -A_3' & A_2\end{pmatrix}C^{-1}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix}^{-1}\eta\right\},$$

$$C = \begin{pmatrix}\Phi'-\Psi'A_1 & \Psi'(E-A_3)\\ \overline{\Psi}'(E-A_3') & \Phi'+\overline{\Psi}'A_2\end{pmatrix}. \qquad (2.11)$$

The form (2.11) is frequently more convenient than (2.9) or (2.9′).

3. In the case of Bose commutation relations the calculation of the functional $\widetilde{A}(\alpha^*,\alpha)$ follows almost verbatim the procedure considered here. The final result is

$$A(\alpha^*,\alpha) = (\det C)^{-1/2}\exp\left[\frac{1}{2}\eta'\begin{pmatrix}A_1 & A_3\\ A_3' & A_2\end{pmatrix}C^{-1}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \Phi\end{pmatrix}^{-1}\eta\right].$$

$$C = \begin{pmatrix}\overline{\Phi}'-\Psi'A_1 & -\Psi'(E+A_3)\\ -\overline{\Psi}'(E+A_3') & \Phi'-\overline{\Psi}'A_2\end{pmatrix}. \qquad (2.12)$$

3. THE THIRRING MODEL

1. The problem solved in the present section consists of the following (see references 2 and 3). We consider the operators $\varphi_1(\xi)$, $\varphi_1^*(\xi)$; $\varphi_2(\xi)$, $\varphi_2^*(\xi)$, satisfying the commutation rules

$$\{\varphi_i(\xi),\ \varphi_j^*(\xi)\} = \delta_{ij}\delta(\xi-\xi'),$$

$$\{\varphi_i(\xi),\ \varphi_j(\xi)\} = \{\varphi_i^*(\xi)\ \varphi_j^*(\xi)\} = 0, \qquad (3.1)$$

and make up from the operators $\varphi_1(\xi)$, $\varphi^*(\xi)$ the operator

$$\hat{V} = \exp\left\{ig\int_{u_0}^{u}\varphi_1^*(\xi)\,\varphi_1(\xi)\,d\xi\right\}. \qquad (3.2)$$

We now go to the Fourier representation

$$\varphi_i(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ip\xi}c_i(p)\,dp, \qquad i=1,\ 2 \qquad (3.3)$$

and then carry out the canonical transformation

$$c_1(p) = \theta_+(p)a(p) + \theta_-(p)\,b^*(-p),$$

$$c_2(p) = \theta_-(p)\,a(p) + \theta_+(p)\,b^*(-p),$$

$$\theta_+(p) = \begin{cases}1 & \text{for } p>0\\ 0 & \text{for } p<0\end{cases}, \qquad \theta_-(p) = \begin{cases}0 & \text{for } p>0\\ 1 & \text{for } p<0\end{cases}. \qquad (3.4)$$

(We leave out the expressions for the conjugate operators).

It is required to write the operator $\hat{V}$ in normal form with respect to a, a*, b, and b*. We shall solve the problem in the following manner. We first write $\hat{V}$ in normal form with respect to $\varphi_1(\xi)$ and $\varphi_1^*(\xi)$. We change to the Fourier representation (3.3) and then carry out canonical transformation (3.4). We see that the infinite factor characteristic of the Thirring model appears only in the last stage.

2. We write $\hat{V}$ in normal form with respect to $\varphi_1(\xi)$ and $\varphi_1^*(\xi)$. For this purpose we expand (3.2) in powers of g and transform each term. We put

$$\hat{A}_n = \int\limits_{u_0\leqslant\xi_i\leqslant u}\varphi_1^*(\xi_1)\,\varphi_1(\xi_1)\ldots\varphi_1^*(\xi_n)\,\varphi_1(\xi_n)\,d\xi_1\ldots d\xi_n, \qquad (3.5)$$

$$\hat{B}_n = \int\limits_{u_0\leqslant\xi_i\leqslant u}\varphi_1^*(\xi_1)\ldots\varphi_1^*(\xi_n)\,\varphi_1(\xi_1)\ldots\varphi_1(\xi_n)\,d\xi_1\ldots d\xi_n. \qquad (3.6)$$

Let us representing $\hat{A}_n$ in the form

$$\hat{A}_n = \sum_{k=0}^{n}c_{nk}\hat{B}_k.$$

We readily find that the c_{nk} satisfy the following recurrence relations

$$c_{nk} = (-1)^{k-1}c_{n-1,\,k-1} + kc_{n-1,\,k},\ c_{11}=1,$$

$$c_{nk} = 0 \text{ for } k<0. \qquad (3.7)$$

From (3.7) we obtain directly

$$c_{n0} = 0, \qquad c_{n1} = 1, \qquad c_{nn} = (-1)^{n-1}c_{n-1,\,n-1}. \qquad (3.8)$$

Further

$$\hat{V} = \sum\frac{(ig)^n}{n!}\hat{A}_n = \sum f_n(g)\,\hat{B}_n, \qquad f_n(g) = \sum_{k=0}^{\infty}\frac{(ig)^k c_{kn}}{k!}.$$

From (3.7) and (3.8) we obtain for $f_k(g)$ the relations

$$f_k'(g) = (-1)^{k-1}f_{k-1}(g) + kf_k(g) \text{ for } k>1;$$

$$f_1(g) = e^{ig}-1, \qquad f_k^{(k)}(0) = (-1)^{k-1}f_{k-1}^{(k-1)}(0). \qquad (3.9)$$

The equations (3.9) are readily solved

$$f_k(g) = (-1)^{k(k-1)/2}(e^{ig}-1)^k/k!. \qquad (3.10)$$

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*It is easy to verify that $\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} = \begin{pmatrix}\overline{\Phi}' & \Psi'\\ \overline{\Psi}' & \overline{\Phi}'\end{pmatrix}$.

We introduce now the anticommuting functions $f_1(\xi)$, $f_2(\xi)$, $f_1^*(\xi)$, and $f_2^*(\xi)$ and set the operator B_n in correspondence with the functional

$$B_n(f_1^*, f_1) = \int\limits_{u_0 \leqslant \xi_i \leqslant u} f_1^*(\xi_1) \ldots f_1^*(\xi_n) f(\xi_1) \ldots f(\xi_n)\, d\xi_1 \ldots d\xi_n = (-1)^{n(n-1)/2} \left(\int\limits_{u_0}^{u} f_1^*(\xi) f_1(\xi)\, d\xi \right)^n .$$

Using (3.10), we find that in this case the operator $\hat{V}$ is set in correspondence with the functional

$$V = \sum \frac{(e^{ig}-1)^n}{n!} \left(\int\limits_{u_0}^{u} f_1^*(\xi)\, f_1(\xi)\, d\xi \right)^n = \exp\left[(e^{ig}-1) \int\limits_{u_0}^{u} f_1^*(\xi)\, f_1(\xi)\, d\xi \right]. \tag{3.11}$$

3. Let us proceed to the Fourier representation. Since in this case $\varphi_i(\xi)$ is expressed only in terms of $c_i(\xi)$, and $\varphi_1^*(\xi)$ only in terms of $c_1^*(\xi)$, this transformation reduces to the following change of variables in (3.11):

$$f_i(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{ip\xi} \sigma_i(p)\, dp,$$

$$f_i^*(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ip\xi} \sigma_i^*(p)\, dp.$$

As a result we have

$$V = \exp\left\{ \iint K(p, p')\, \sigma_1^*(p)\, \sigma_1(p')\, dp\, dp' \right\}, \tag{3.12}$$

$$K(p, p') = \frac{g'}{2\pi i} \, \frac{e^{i(p'-p)u} - e^{i(p'-p)u_0}}{p'-p}, \qquad g' = e^{ig} - 1. \tag{3.13}$$

The rest of the problem consists of transforming the functional (3.12). We thus encounter the situation considered in the preceding section. The solution is given by formulas (2.9) and (2.11). We solve, however, a somewhat more general problem, namely, we find the canonical transformation of (3.4) of the functional (3.12) with an arbitrary kernel $K(p, p')$.

4. In order to make use of formula (2.11), we must first find the operator C [see (2.11)], the operator C^{-1}, and det C. We note first that the canonical transformation (3.4) is given with the aid of the matrix kernel

$$\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}; \quad \Phi = \begin{pmatrix} \theta_+(p) & 0 \\ \theta_-(p) & 0 \end{pmatrix} \delta(p - p'),$$

$$\Psi = \begin{pmatrix} 0 & \theta_-(p) \\ 0 & \theta_+(p) \end{pmatrix} \delta(p + p'). \tag{3.14}$$

The operator $\mathcal{A}$ [see (2.3)] is given by the matrix kernels A_1, A_2, and A_3, which are equal to

$$A_1 = A_2 = 0, \quad A_3 = \begin{pmatrix} -K(p, p') & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.15}$$

A simple calculation shows that the operator C is given by the matrix kernel

$$C = C(p, p') = \begin{pmatrix} \overline{\Phi}' & \Psi'(E - A_3) \\ \overline{\Psi}'(E - A_3') & \Phi' \end{pmatrix}, \tag{3.16}$$

where

$$\Phi' = \overline{\Phi}' = \begin{pmatrix} \theta_+(p) & \theta_-(p) \\ 0 & 0 \end{pmatrix} \delta(p - p'), \qquad K'(p, q) = K(q, p),$$

$$\Psi'(E - A_3) = \begin{pmatrix} 0 & 0 \\ \theta_+(p) & \theta_-(p) \end{pmatrix} \delta(p + p') + \begin{pmatrix} 0 & 0 \\ \theta_+(p) K(-p, p') & 0 \end{pmatrix},$$

$$\overline{\Psi}'(E - A_3') = \Psi'(E - A_3') = \begin{pmatrix} 0 & 0 \\ \theta_+(p) & \theta_-(p) \end{pmatrix} \delta(p + p') + \begin{pmatrix} 0 & 0 \\ \theta_+(p) K'(-p, p') & 0 \end{pmatrix}. \tag{3.16$'$}$$

5. Let us determine the operator C^{-1}. For this purpose we solve the system of equations $Cf = g$ (f and g are columns consisting of four functions: $f = \begin{pmatrix} f_1 \\ \vdots \\ f_4 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ \vdots \\ g_4 \end{pmatrix}$).

Using (3.16) and (3.16′), we write this system out in detail

$$\begin{aligned} &\theta_+(p) f_1(p) + \theta_-(p) f_2(p) = g_1(p), \\ &\theta_+(p) f_3(-p) + \theta_+(p) \int K(-p, p') f_3(p')\, dp' + \theta_-(p) f_4(-p) = g_2(p), \\ &\theta_+(p) f_3(p) + \theta_-(p) f_4(p) = g_3(p), \\ &\theta_+(p) f_1(-p) + \theta_+(p) \int K'(-p, p') f_1(p')\, dp' + \theta_-(p) f_2(-p) = g_4(p). \end{aligned} \tag{3.17}$$

The system (3.17) breaks up in a way that the first and fourth and the second and third equations form independent systems, which differ from each other only in that K is replaced by K' and the numbering of the functions f_i and g_i is different. It is sufficient therefore to solve the system consisting of the second and third equations:

$$\theta_+(p) f_3(p) + \theta_-(p) f_4(p) = g_3(p), \tag{3.18$'$}$$

$$\theta_+(p) f_3(-p) + \theta_+(p) \int K(-p, p') f_3(p')\, dp' + \theta_-(p) f_4(-p) = g_2(p). \tag{3.18$''$}$$

In (3.18″) we replace p by $-p$

$$\theta_-(p) f_3(p) + \theta_-(p) \int K(p, p') f_3(p')\, dp' + \theta_+(p) f_4(p) = g_2(-p). \tag{3.18$'''$}$$

We multiply (3.18′) by $\theta_+(p)$ and (3.18‴) by $\theta_-(p)$ and add. We obtain

$$f_3(p) + \theta_-(p) \int K(p, p') f_3(p')\, dp' = \theta_+(p) g_3(p) + \theta_-(p) g_2(-p).$$

We write the solution of this equation in the form

$$f_3(p) = \int G(p, p')\,[\theta_+(p')g_3(p') + \theta_-(p')\,g_2(-p')]\,dp'. \tag{3.19}$$

Here G(p, p′) is the kernel of the operator $\hat{G}$, the inverse of the operator $(E - \theta_-\hat{K})$ given by the kernel $\delta(p - p') + \theta_-(p)K(p, p')$. Let us multiply, finally, (3.18′) by $\theta_-(p)$ and (3.18‴) by $\theta_+(p)$ and add. We have

$$f_4(p) = \theta_-(p)\,g_3(p) + \theta_+(p)\,g_2(-p). \tag{3.20}$$

We solve analogously the system comprising the first and fourth equations. As a result we have

$$f_1(p) = \int \widetilde{G}(p, p')\,[\theta_+(p')\,g_1(p') + \theta_-(p')\,g_4(-p')]\,dp'.$$

$$f_2(p) = \theta_-(p)\,g_1(p) + \theta_+(p)\,g_4(-p), \tag{3.21}$$

where $\widetilde{G}(p, p')$ is the kernel of the operator $\hat{\widetilde{G}}$, the inverse of the operator $(E - \theta_-\hat{K}')$ specified by the kernel $\delta(p - p') + \theta_-(p)K'(p, p')$.

From (3.19), (3.20), and (3.21) we find that the operator C^{-1} is given by the matrix kernel

$$C^{-1}(p, p') = \begin{pmatrix} \widetilde{G}(p, p')\,\theta_+(p') & 0 & 0 & \widetilde{G}(p, -p')\theta_+(p') \\ \theta_-(p)\,\delta(p - p') & 0 & 0 & \theta_+(p)\,\delta(p + p') \\ 0 & G(p, -p')\,\theta_+(p') & G(p, p')\,\theta_+(p') & 0 \\ 0 & \theta_+(p)\,\delta(p + p') & \theta_-(p)\,\delta(p - p') & 0 \end{pmatrix} \tag{3.22}$$

6. In order to calculate the exponent in (2.11), we must find the operator

$$M = \mathcal{A}C^{-1}\begin{pmatrix} \overline{\Phi}' & \Psi' \\ \overline{\Psi}' & \Phi' \end{pmatrix}$$

The operators A, C^{-1}, Φ, and Ψ are given by formulas (3.14), (3.15), and (3.22). The calculations entail no difficulties. As a result we obtain

$$M = \begin{pmatrix} 0 & 0 & -\hat{K}\hat{G} & 0 \\ 0 & 0 & 0 & 0 \\ \hat{K}'\hat{\widetilde{G}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

It is easy to verify that $\hat{K}'\hat{\widetilde{G}} = (\hat{K}\hat{G})'$. Therefore $M = -M'$ (see Sec. 1, Item 2). Thus we find that the transformed functional has the form

$$\widetilde{V} = (\det C)^{1/2}\exp\left\{\int L(p, q)\,y^*(p)\,y(q)\,dp\,dq\right\}, \tag{3.24}$$

where L(p, q) is the kernel of the operator $\hat{K}\hat{G}$, $y(p) = \theta_+(p)\,\alpha(p) + \theta_-(p)\,\beta^*(-p)$. Here $\alpha(p)$, $\beta(p)$, $\alpha^*(p)$, and $\beta^*(p)$ are functions with anticommuting values, corresponding to the operators a(p), b(p), a*(p), and b*(p).

7. Let us calculate det C. For this purpose we interchange the second and fourth rows in the expressions (3.16), (3.16′) for C. As a result we obtain a matrix $\widetilde{C}$ in the form

$$\widetilde{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \theta_+(p)\;\delta(p - p') & \theta_-(p)\,\delta(p - p') \\ \theta_+(p)\;(\delta(p + p') - K(-p, p')) & \theta_-(p)\,\delta(p + p') \end{pmatrix}, \tag{3.25}$$

C_1 differs from C_2 in that K is replaced by K′.

It is obvious that det C = ± det $\widetilde{C}$. A more detailed analysis* shows that det C = det $\widetilde{C}$. Let us find det C_2. Since the determinant is a product of the eigenvalues, it is necessary to find the eigenvalues of the system

$$\theta_+(p)\,x(p) + \theta_-(p)\,y(p) = \lambda x(p), \tag{3.26′}$$

$$\theta_+(p)\,x(-p) + \theta_+(p)\int K(-p, p')\,x(p')\,dp' + \theta_-(p)\,y(-p) = \lambda y(p). \tag{3.26″}$$

We replace p by −p in (3.26″)

$$\theta_-(p)\,x(p) + \theta_-(p)\int K(p, p')\,x(p')\,dp' + \theta_+(p)\,y(p) = \lambda y(-p) \tag{3.26‴}$$

and multiply (3.26′) by $\theta_+(p)$

$$\theta_+(p)\,x(p) = \lambda\,\theta_+(p)\,x(p).$$

From this we get either that λ = 1 or that $x(p) = \theta_-(p)\widetilde{x}(p)$.

Let us consider the second possibility. We substitute $x(p) = \theta_-(p)\widetilde{x}(p)$ in (3.26‴):

$$\theta_-(p)\widetilde{x}(p) + \theta_-(p)\int K(p, p')\,\theta_-(p')\,\widetilde{x}(p')\,dp' + \theta_+(p)\,y(p) = \lambda y(-p). \tag{3.27}$$

In order to eliminate y, we multiply (3.26‴) by $\theta_+(p)$:

$$\theta_+(p)\,y(p) = \lambda\theta_+(p)\,y(-p). \tag{3.28}$$

From this, by reversing the sign of p, we obtain

$$\theta_-(p)\,y(-p) = \lambda\theta_-(p)\,y(p).$$

Combining this relation with (3.26′) and with $x(p) = \theta_-(p)\widetilde{x}(p)$, we obtain

$$\lambda^{-1}\theta_-(p)\,y(-p) = \lambda\theta_-(p)\,\widetilde{x}(p).$$

Hence

$$\theta_-(p)\,y(-p) = \lambda^2\theta_-(p)\,\widetilde{x}(p). \tag{3.29}$$

Ultimately, multiplying (3.27) by $\theta_-(p)$ and combining with (3.29), we obtain

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*Let us consider the function $K_\alpha(p, q)$, which is continuous in α and such that $K_1(p, q) = K(p, q)$ and $K_0(p, q) = 0$. It is obvious that det C = D(α) is a continuous function of α and D(0) = 1. The last condition is satisfied only if det C = det $\widetilde{C}$.

$$\theta_-(p)\,\widetilde{x}(p)+\theta_-(p)\int K(p,p')\,\theta_-(p')\,\widetilde{x}(p')\,dp' = \lambda^3\theta_-(p)\,\widetilde{x}(p). \qquad (3.30)$$

Thus, the cubes of the eigenvalues of C_2 that are different from unity are the eigenvalues of Eq. (3.30).

Let now $\mu \neq 0$, $\mu \neq 1$ be the eigenvalue of Eq. (3.30), and let λ_1, λ_2, and λ_3 be the different cube roots of μ. The formulas (3.29) and (3.28) make it possible to construct for each λ_i a function $y_i(p)$, and obviously $y_i \neq y_j$ if $\lambda_i \neq \lambda_j$. Thus, we obtain finally that to each eigenvalue $\mu \neq 1$, $\mu \neq 0$ of Eq. (3.30) correspond three different roots $\lambda_i = (\mu)^{1/3}$ of the operator C_2 and the roots obtained account for all the eigenvalues of C_2. Therefore finally

$$\det C_2 = \det(E + K_-), \qquad (3.31)$$

where $E + K_-$ is an operator specified on the semi-axis $p < 0$ by the kernel $\delta(p-k) + K(p,q)$. Analogously, $\det C_1 = \det(E - (K')_-)$. Since obviously $(K')_- = (K_-)'$, we have $\det C_1 = \det C_2$. Therefore, finally,

$$(\det C)^{1/2} = \det(E + K_-). \qquad (3.32)$$

Formulas (3.24) and (3.32) give the complete solution of the problem. Substituting K from (3.13) in (3.32) we find that $(\det C)^{1/2} = \infty$.

4. CONCLUSION

The result obtained in the present paper agrees with the result of Glaser[3] and differs somewhat from the results of Thirring.[2] The difference lies in the fact that both in Glaser's paper and in ours the final answer depends on the constant $g' = e^{ig} - 1$, where g is the interaction constant in the initial Hamiltonian. In order to obtain from the final formulas (3.24) and (3.32) the corresponding Thirring formulas, it is necessary to replace everywhere $g' = e^{ig} - 1$ by $g'' = ig/(1 - ig)$.

The disagreement between Glaser's and Thirring's results is due to definite mathematical factors which will be discussed elsewhere.

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ANALYSIS OF THE ANGULAR DISTRIBUTION OF REACTION PRODUCTS IN LIGHT NUCLEI

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Measurement of the angular distributions at energies $E_\pm = E_r \pm \frac{1}{2}\Gamma_r$ and $E_0 = E_r$, where E_r and Γ_r are the isolated resonance energy and width, respectively, permits one to separate the contribution of the direct mechanism to the reaction cross section.

IN the (dp), (dn), (pn) and other reactions involving light nuclei and incident particles of medium energy, isolated resonances corresponding to quasi-discrete levels of a compound nucleus are observed in a number of cases.[1] However, the angular distributions of the reaction products are not usually symmetric with respect to 90°, and have maxima which are characteristic for direct interaction processes. This gives a basis for the assumption that direct and resonance mechanisms for the reaction are both present simultaneously. Then, owing to the interference between the two mechanisms, an analysis of experimental data on the angular distributions is extremely difficult. In the present article, it is shown that if the energy of the incident particles is sufficiently large in comparison with the height of the Coulomb barrier, then the measurement of the angular distribution at energies E_r and $E_r \pm \frac{1}{2}\Gamma_r$ (where E_r and Γ_r are the energy and width of the isolated resonance) permits us to separate the contribution of the direct mechanism to the differential cross section for the reaction.

The reaction amplitude in the vicinity of an isolated resonance of a compound nucleus can be written in the form

$$f = f_d + f_r,$$

$$f_d = -\frac{m_f}{2\pi\hbar^2}\langle\Phi_f|\hat{V}_d|\Phi_i\rangle,$$

$$f_r = -\frac{m_i}{2\pi\hbar^2}\frac{\langle\Phi_f|\hat{V}_f|X_r\rangle\langle X_r|\hat{V}_i|\Phi_i\rangle}{E - E_r + i\Gamma_r/2}, \qquad (1)$$

where Φ_i and Φ_f are the wave functions of the initial and final states of the system, $\hat{V}_d$ is the direct-interaction operator, $\hat{V}_i$ and $\hat{V}_f$ are the interactions in the initial and final states, X_r is the wave function of the compound nucleus, and E_r and Γ_r are the energy and width of the resonance.

For incident particle energies greater than the Coulomb barrier (in the region of carbon $E_c = Ze^2/R \sim 3$ Mev), the angular distributions of the direct reactions are satisfactorily described in the plane-wave approximation.[2] In this approximation, in the simpler case of inelastic scattering of spin-zero particles, the direct mechanism amplitude is

$$f_d = -\frac{m}{2\pi\hbar^2}\sum_L i^L\sqrt{4\pi(2L+1)}\,C^{j_f m_f}_{L0;\,j_i m_i}F^L_{j_i j_f}(K_{if}),$$

$$K_{if} = \sqrt{k_i^2 + k_f^2 - 2k_i k_f\cos\theta}, \qquad k_{i,f} = \sqrt{2m\varepsilon_{if}}/\hbar, \qquad (2)$$

where j_i and j_f are the spins of the initial and final nuclei, L is the transferred orbital angular momentum, and θ is the c.m.s. scattering angle. The nuclear matrix elements $F^L_{j_i j_f}(K_{if})$ of $\hat{V}_d$ between Φ_f and Φ_i determine the differential cross section for the direct mechanism of the reaction:

$$\frac{d\sigma_d}{d\Omega} = \frac{2j_f+1}{2j_i+1}\frac{m^2}{\pi\hbar^4}\frac{k_f}{k_i}\sum_L|F^L_{j_i j_f}(K_{if})|^2.$$

The resonance mechanism amplitude is equal to

$$f_r = -\frac{2m}{\hbar^2}\frac{1}{E - E_r + i\Gamma_r/2}\sum_{l\lambda\mu} i^{l-\lambda}\sqrt{4\pi(2l+1)}\,Y_{\lambda\mu}(\theta,\varphi)$$
$$\times\, v^l_{j_i I}v^\lambda_{j_f I}C^{IM}_{l0;\,j_i m_i}C^{IM}_{\lambda\mu;\,j_f m_f}, \qquad (3)$$

where l and λ are the orbital angular momenta of the partial waves in the incoming and outgoing channels, I is the spin of the resonance level of the compound nucleus. The nuclear matrix elements $v^l_{j_i I}$ and $v^\lambda_{j_f I}$ of $\hat{V}_i$ and $\hat{V}_f$ between Φ_i and Φ_f, and X_r determine the partial widths of the incoming and outgoing channels:

$$\Gamma^l_{j_i I} = 4m_i k_i\hbar^{-2}|v^l_{j_i I}|^2, \qquad \Gamma^\lambda_{j_f I} = 4m_f k_f\hbar^{-2}|v^\lambda_{j_f I}|^2.$$

The direct and resonance mechanism amplitudes in stripping reactions are of similar form (see, for example, reference 3).

The interference term in the differential cross section, as seen from (2) and (3), contains the factor

$$\mathrm{Re}\,\frac{i^{L+l+\lambda}F^{L*}_{j_i j_f}v^{l}_{j_i I}v^{\lambda}_{j_f I}}{E-E_r+i\Gamma_r/2}\,\delta_{\mu 0}.$$

Owing to the law of conservation of parity, the sum $L + l + \lambda$ is always even, regardless of the spin of the particles taking part in the reaction. Therefore the interference term close to E_r has the form

$$\frac{d\sigma_{int}}{d\Omega}=J^i_1+J^i_2,\qquad J^i_1=\frac{E-E_r}{(E-E_r)^2+\Gamma_r^2/4}\sum_{Ll\lambda}g^L_{l\lambda}(\theta)\,A,$$

$$J^i_2=\frac{{}^1/{}_2\Gamma_r}{(E-E_r)^2+\Gamma_r^2/4}\sum_{Ll\lambda}g^L_{l\lambda}(\theta)\,B, \tag{4}$$

where we have introduced the notation

$$F^{L*}_{j_i j_f}v^{l}_{j_i I}v^{\lambda}_{j_f I}\equiv A+iB,$$

and $g^L_{l\lambda}(\theta)$ are real functions of the scattering angle. The term J^i_1, containing A, vanishes for $E = E_r$, its absolute value attains a maximum at $|E - E_r| = {}^1/{}_2\,\Gamma_r$, and changes sign upon passing through resonance. The term J^i_2, containing B, depends on the energy in the same way as the resonance cross section

$$\frac{d\sigma_r}{d\Omega}=\frac{G_r(\theta)}{(E-E_r)^2+\Gamma_r^2/4},\qquad G_r(\theta)=G_r(\pi-\theta). \tag{5}$$

For incident particles of medium energy, the experimentally observed resonance widths in light nuclei are of the order 10 — 100 kev. In the region far from the reaction threshold, the direct mechanism cross section $d\sigma_d/d\Omega$ varies appreciably over intervals of the order of 1 — 2 Mev and is practically constant within the limits of the resonance width. Denoting

$$J(E;\theta)=d\sigma_d/d\Omega+d\sigma_r/d\Omega+d\sigma_{int}/d\Omega,$$

we readily obtain from formulas (4) and (5)

$$d\sigma_d/d\Omega=J(E_r+\tfrac{1}{2}\Gamma_r;\,\theta)+J(E_r-\tfrac{1}{2}\Gamma_r;\,\theta)-J(E_r;\,\theta). \tag{6}$$

Hence the proposed method of measurement of the angular distributions makes it possible to separate the contribution of the direct mechanism to the reaction cross section from the compound nucleus formation mechanism.

It is also of interest to determine the sum $d\sigma_r/d\Omega + J^i_2 = \mathscr{K}_r(\theta)$, characterizing the contribution from the compound nucleus formation mechanism to the angular distribution. According to (4) and (5),

$$\mathscr{K}_r(\theta)=2J(E_r;\,\theta)-J\left(E_r+\tfrac{1}{2}\Gamma_r;\,\theta\right)-J\left(E_r-\tfrac{1}{2}\Gamma_r;\,\theta\right).$$

Since part of the interference term J^i_2 occurs in $\mathscr{K}_r(\theta)$, then $\mathscr{K}_r(\theta)$ is not symmetric with respect to 90° if $B \neq 0$. If $B = 0$, then $\mathscr{K}_r(\theta) = \mathscr{K}_r(\pi - \theta)$, since in this case $\mathscr{K}_r(\theta) = (d\sigma_r/d\Omega)|_{E = E_r}$. If this condition is fulfilled, analysis of the angular distribution permits one to determine the spin of the resonance level of the compound nucleus.

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ABSORPTION OF ULTRASOUND IN AN ANISOTROPIC SUPERCONDUCTOR

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Anisotropy of superconductors is taken into account within the framework of the Gor'kov method.[1] The electron-phonon interaction is not considered small, but the smallness of the sound velocity in comparison with the Fermi velocity is explicitly taken into account. It is shown that the low-temperature region $T \ll T_c$ is divided into two regions in which the behavior of the sound attenuation as a function of direction and temperature is significantly different. Sound absorption exhibits some peculiarities at low temperatures in semiconductors possessing Fermi surfaces of the "corrugated" type. The problem of the possibility of determining the energy gap from data on the attenuation of ultrasound is analyzed.

1. INTRODUCTION

RECENTLY several works have appeared which are devoted to the study of the effects of anisotropy in superconductors. There is first the theoretical work of Khalatnikov,[2] which is devoted to the anisotropy of heat conduction, and the experimental researches on the measurement of heat capacity,[3] heat conduction[4] and ultrasonic absorption.[5,6]

The natural purpose of the experimental researches was the determination of the energy gap Δ as a function of direction. However, as theory shows,[2] at temperatures T much less than the transition temperature T_c, the thermal conductivity is determined only by the minimum value of the energy gap, while a significant anisotropy in the thermal conductivity arises only in the case in which there are two minima for Δ (in opposite directions). Therefore, information on Δ obtained from experimental data on thermal conductivity is very limited.

So far as research on the measurement of ultrasonic absorption is concerned,[5,6] the authors therein, in their treatment of the experimental data, made use of the formula for the ratio α_s/α_n, introduced by Bardeen, Cooper and Schrieffer (denoted below as BCS) for the isotropic case (α_s, α_n — absorption coefficients of ultrasound in the superconducting and normal states, respectively):

$$\alpha_s/\alpha_n = 2f(\beta\Delta), \qquad f(x) = (e^x + 1)^{-1}, \qquad \beta = 1/T\,. \quad (1)$$

As will be shown in our work, Eq. (1) is not valid in the anisotropic case. The quantity $f(\beta\Delta)$ appears in the correct formula; it is averaged with some weight of very complicated form along the line (on the Fermi surface) on which the Fermi velocity is perpendicular to the direction of sound propagation. Therefore, experiments on the ultrasonic attenuation generally give incomplete information on the function $\Delta(\mathbf{n})$, although the results are much more fruitful than the measurements of thermal conductivity.

Moreover, Eq. (1) at low temperatures does not reflect a number of peculiarities which are properties of the anisotropic superconductors. Although the ultrasonic absorption in a given direction actually falls off exponentially with temperature, it is shown that the exponent, generally speaking, depends on the direction. At very low temperatures (see Sec. 4), the exponent does not depend on the direction but then the absorption has anomalies along certain directions.

Finally, the approximation made in the work of BCS (the smallness of the interaction constant between electrons) is found to be insufficient for the correct description of sound absorption in the region of low temperatures. In this connection, we shall construct a theory in which the quantity c/v_F is used as a small parameter (c — sound velocity, v_F — velocity of the electron on the Fermi surface).

2. THE ELECTRON-PHONON INTERACTION GREEN'S FUNCTION

We shall limit ourselves to consideration of the interaction of electrons with phonons, which we shall describe by a Hamiltonian of the Fröhlich type:[7]

$$H = \sum_{s,\,\mathbf{p},\,\mathbf{q}} \gamma_s(\mathbf{p}, \mathbf{q})\, a^+_{\mathbf{p}} a_{\mathbf{p}+\mathbf{q}} b^+_{s\mathbf{q}} + \text{Herm. conj.} \quad (2)$$

where a_p^+, a_p are the creation and annihilation operators of electrons and holes with momentum $\mathbf{p}$: $b_{s\mathbf{q}}^+$ is the creation operator of a phonon with momentum $\mathbf{q}$ and polarization s ($s = 1, 2, 3$); $\gamma_s(\mathbf{p}, \mathbf{q})$ is the matrix element of the energy of interaction. For sufficiently small $\mathbf{q}$ ($q \ll p_0 \sim \pi/a$, where a is the lattice constant) $\gamma_s(\mathbf{p}, \mathbf{q})$ can be represented in the form*

$$\gamma_s(\mathbf{p}, \mathbf{q}) = \beta_s(\mathbf{q})\, g_0(\mathbf{p}), \quad \beta_s^2(\mathbf{q}) = \pi^2 \lambda_s(\mathbf{n})\, \omega_s^0(\mathbf{q}) / p_0, \qquad \mathbf{n} = \mathbf{q}/q. \tag{3}$$

Here, $\omega_s^0(\mathbf{q})$ — frequency of a phonon with momentum $\mathbf{q}$ and polarization s; $\lambda_s(\mathbf{n})$ — nondimensional function of the interaction. As usual, $\omega_s^0(\mathbf{q}) = c_s(\mathbf{n})q$, where $c_s(\mathbf{n})$ — sound velocity, $g_0(\mathbf{p})$ — some dimensionless function which describes the anisotropy of the interaction.

Following Matsubara,[8] we introduce the Green's function (more precisely, tensor) of the phonons:

$$D_{ss'}(\mathbf{q}, \tau - \tau') = i \langle T_\tau b_s(\mathbf{q}, \tau)\, b_s^+(\mathbf{q}, \tau') \rangle\, \beta_s(\mathbf{p})\, \beta_{s'}(\mathbf{q}), \tag{4}$$

where $\langle A \rangle$ denotes the trace of $e^{-\beta(H-\mu N)}A$ and

$$b_s(\mathbf{p}, \tau) = e^{\tau H} b_s(\mathbf{p})\, e^{-\tau H}, \qquad b_s^+(\mathbf{p}, \tau) = e^{\tau H} b_s^+(\mathbf{p})\, e^{-\tau H}.$$

As Gor'kov has shown,[1] in the case of a superconductor the electrons are described by two Green's functions G and F, which we shall define as:

$$G(\mathbf{p}, \tau - \tau') = i \langle T_\tau a(\mathbf{p}, \tau)\, a^+(\mathbf{p}, \tau') \rangle\, g_0(\mathbf{p}), \tag{5}$$

$$F(\mathbf{p}, \tau - \tau') = \langle N | e^{\beta(H-\mu N)} T a^+(\mathbf{p}, \tau)\, a^+(-\mathbf{p}, \tau') | N + 2 \rangle\, g_0(\mathbf{p}). \tag{6}$$

In Eq. (6), $\langle N|$, $|N + 2\rangle$ are the microcanonical states with N and $N + 2$ particles, corresponding to the given temperature.

In what follows we shall transform from the "time" representation of the functions, D, G, F [Eqs. (4), (5), (6)] to the "representation of imaginary frequencies," developed in references 9 and 10. We shall use for construction of the functions $D_{ss'}(\mathbf{q}, i\omega_n)$, $G(\mathbf{p}, i\eta_n)$, and $F(\mathbf{p}, i\eta_n)$ (where $\omega_n = 2n\pi T$, $\eta_n = (2n + 1)\pi T$), a diagram technique in which the "zero" Green's functions correspond to the lines:

$$D_{ss'}^0 = \delta_{ss'} D_s, \quad D_s(\mathbf{q}, i\omega_n) = 2\beta_s^2(\mathbf{q})\, \omega_s^0(\mathbf{q}) / [\omega_s^{02}(\mathbf{q}) + \omega_n^2], \tag{7}$$

$$G^0(\mathbf{p}, i\eta_n) = g_0(\mathbf{p}) / (\xi_\mathbf{p}^0 - i\eta_n), \tag{8}$$

$$F^0(\mathbf{p}, i\eta_n) = 0, \tag{9}$$

where, as usual,

$$\xi_\mathbf{p} = \mathbf{v}_F^0(\mathbf{n})\, (\mathbf{p} - \mathbf{p}_F). \tag{10}$$

Here $\mathbf{v}_F$ is considered to be dependent on the direction; the Fermi surface $\xi_\mathbf{p} = 0$ represents some arbitrary surface in momentum space. For a given choice of "zero" Green's functions, the elementary vertex in the diagram corresponds to the factor $-i$.

We introduce the polarization operator Π in the usual way:

$$D_{ss'} = D_{ss'}^0 + \sum_{r, r'} D_{sr}^0 \Pi_{rr'} D_{r's}. \tag{11}$$

It is obvious that in our definition of the Green's functions (4) and (7) — (9) the operator $\Pi_{rr'}$ does not depend on the indices r, r':

$$\Pi_{rr'} = \Pi. \tag{12}$$

Substituting (12) in (11), and solving the complete set of equations relative to $D_{ss'}$, we get

$$D_{ss'} = \delta_{ss'} D_s + D_s D_{s'} \Pi / (1 - D\Pi), \qquad D = \sum_s D_s. \tag{13}$$

The frequency and damping of the acoustic vibrations are determined by the poles $\omega = \omega_s(\mathbf{q})$ of the functions $D_{ss'}(\mathbf{q}, \omega)$, which are analytically continued with discrete imaginary values $\omega = i\omega_n$ in the complex plane ω. We recall that $D_{ss'}(\mathbf{q}, \omega)$ should be analytic in the upper half-plane of ω; the poles $\omega_s(\mathbf{q})$ should located in the lower half-plane close to the real axis. Obviously, the poles $D_{ss'}(\mathbf{q}, \omega)$ coincide with the roots of the equation

$$1 - D\Pi = 0. \tag{14}$$

We emphasize that the poles D_s are not the poles $D_{ss'}$ (for $\Pi \neq 0$).

The Dyson equations for the functions D, G, F can be found in the work of the author.[11] In the equations described there [Eqs. (3) — (8)] obvious changes should be made corresponding to the transition from integration over the frequency to summation, and also corresponding to summation over the polarizations.

It can be shown (compare reference 12) that in the regions of interest to us ($\omega \sim cq$) the complete vertex part Γ_1 can be replaced by unity, with accuracy up to a small quantity $\sim c/v_F$, and Γ_2 by zero.

This makes it possible to apply the method developed by Migdal for the solution of the Dyson equations[12] (see also reference 13). However, we have a direct interest only in the attenuation of the ultrasound. Therefore we shall not be concerned here with the exact solution of the Dyson equations, but shall note only some properties of G and F, which are necessary in what follows. We write down the Dyson equations for G and F:

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*It would have been more accurate to write $\gamma_s(\mathbf{p}, \mathbf{q})$ in the form of a sum of products of the type (3), which would have greatly complicated the calculations without changing the results.

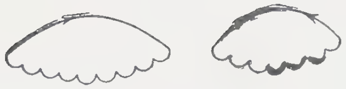

FIG. 1

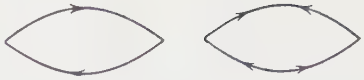

FIG. 2

$$G(p) = G_0(p) + [G_0(p)\,\Sigma_1(p)\,G(p) - G_0(p)\,\Sigma_2(p)\,F(p)]\,g_0^{-1}(\mathbf{p}), \tag{15}$$

$$F(p) = [G_0(-p)\,\Sigma_1(-p)\,F(p) + G_0(-p)\,\Sigma_2(p)\,G(p)]\,g_0^{-1}(\mathbf{p}), \tag{16}$$

where p is the 4-vector $(\mathbf{p}, i\eta_n)$. The operators of the "self energy" Σ_1, Σ_2 in the given approximation are determined by the diagrams of Fig. 1, where the smooth lines correspond to the complete Green's functions.

First we note that the values $\xi_p \gg \Delta$ play a role in Σ_1, and therefore $\Sigma_1(p)$ has practically the same value as in a normal metal. Furthermore, for values of T of interest to us (below the degeneracy temperature) the function $\Sigma_1(p)$ is almost independent of T, and we can replace it by the value at zero temperature. We now introduce the function $G_M(p)$

$$G_M(p) = G_0(p)\,[1 - g_0^{-1}(\mathbf{p})\,G_0(p)\,\Sigma_1(p)]^{-1}. \tag{17}$$

The function $G_M(p)$ is identical with that found by Migdal[12] for the normal metal (naturally, with account of anisotropy and replacement of η for $i\eta_n$). In reference 12 it was shown (the proof and result are only slightly changed upon consideration of anisotropy) that $G_M^{-1}(p)$ has the form

$$G_M^{-1}(p) = (\xi_{\mathbf{p}} - \varphi(\mathbf{n};\eta))\,g_0^{-1}(\mathbf{p}), \quad \mathbf{n} = \mathbf{p}/p, \tag{18}$$

where $\varphi(\mathbf{n}, \eta)$ is an odd function of η, while $\operatorname{Im}\varphi \ll \operatorname{Re}\varphi$ both for $\eta \ll \omega_0$ and for $\eta \gg \omega_0$ (ω_0 – Debye frequency) [the factor $g_0^{-1}(\mathbf{p})$ is, of course, absent in reference 12].

Solving the system (15), (16) with account of the definition (17), we obtain

$$G(\mathbf{p}, \eta) = \left(\frac{u_{\mathbf{p}}^{02}}{\varepsilon_{\mathbf{p}}^0 - \varphi(\mathbf{n}, \eta)} - \frac{v_{\mathbf{p}}^{02}}{\varepsilon_{\mathbf{p}}^0 + \varphi(\mathbf{n}, \eta)}\right) g_0(\mathbf{p}), \tag{19}$$

$$F(\mathbf{p}, \eta) = \frac{\Sigma_2}{2\varepsilon_{\mathbf{p}}^0}\left(\frac{1}{\varepsilon_{\mathbf{p}}^0 - \varphi(\mathbf{n}, \eta)} + \frac{1}{\varepsilon_{\mathbf{p}}^0 + \varphi(\mathbf{n}, \eta)}\right) g_0(\mathbf{p}), \tag{20}$$

where

$$u_{\mathbf{p}}^{02} = \tfrac{1}{2}(1 + \xi_{\mathbf{p}}^0/\varepsilon_{\mathbf{p}}^0), \quad v_{\mathbf{p}}^{02} = \tfrac{1}{2}(1 - \xi_{\mathbf{p}}^0/\varepsilon_{\mathbf{p}}^0), \quad \varepsilon_{\mathbf{p}}^{02} = \xi_{\mathbf{p}}^{02} + \Sigma_2^2, \tag{21}$$

while the dependence of Σ_2 on $|\mathbf{p}|$ and η can be neglected.

The energy of the electron in the superconductor is determined by the roots of the equation

$$\varphi(\eta) = \varepsilon_{\mathbf{p}}^0. \tag{22}$$

We denote the solution of this equation by ϵ_{p}. Equation (22) reduces to a certain renormalization of the quantities ξ_p and Σ_2 (we emphasize that the renormalization of ξ_p and Σ_2 is the same). Therefore, we can write Eqs. (19), (20) in the form

$$G(\mathbf{p}, \eta) = \left(\frac{u_{\mathbf{p}}^2}{\varepsilon_{\mathbf{p}} - \eta} - \frac{v_{\mathbf{p}}^2}{\varepsilon_{\mathbf{p}} + \eta}\right) g(\mathbf{p}) + A(\mathbf{p}, \eta),$$

$$F(\mathbf{p}, \eta) = \frac{\Delta(\mathbf{p})}{2\varepsilon_{\mathbf{p}}}\left(\frac{1}{\varepsilon_{\mathbf{p}} - \eta} + \frac{1}{\varepsilon_{\mathbf{p}} + \eta}\right) g(\mathbf{p}) + B(\mathbf{p}, \eta). \tag{23}$$

The functions without the index refer to the renormalized quantities; $\Delta(\mathbf{p})$ (the energy gap) is the renormalized Σ_2. The functions $A(\mathbf{p}, \eta)$, $B(\mathbf{p}, \eta)$ no longer have divergencies and are single-valued in the complex plane η with two cuts along the real axis $(\pm\Delta, \pm\infty)$.[11, 13] However, in the approximation which we have made we have, for real η, $\operatorname{Im} A = \operatorname{Im} B = 0$, while $\operatorname{Re} A$ and $\operatorname{Re} B$ are single-valued functions of η.

3. THE ATTENUATION OF THE ULTRASOUND

We shall make a detailed study of the function $\Pi(\mathbf{p}, i\omega_n)$ which is defined by the diagram of Fig. 2:

$$\Pi(\mathbf{q}, i\omega_n) = \frac{T}{(2\pi)^3}\sum_{\eta_n}\int d^3p\,[G(p)\,G(p-q) - F(p)\,F(p-q)]. \tag{24}$$

It is convenient to go from summation over η_n to integration by means of

$$T\sum_{\eta_n}\cdots \to \frac{i}{4\pi}\int_\Gamma d\eta\,\operatorname{th}\frac{\beta\eta}{2}\cdots, \tag{25*}$$

where Γ is a contour consisting of two straight lines parallel to the imaginary axis η. In integration over η, we deform the contour Γ so that there remain residues at the poles $\eta = \pm\epsilon_{\mathrm{p}}$ or integrals over the cuts. By virtue of the remark made in Sec. 2, the integrals over the cuts vanish in our approximation. Further, components appear of the type

$$\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\,g(\mathbf{p})\,u_{\mathbf{p}}^2 A(\mathbf{p} - \mathbf{q}, \varepsilon_{\mathbf{p}} - \omega). \tag{26}$$

The integration here is effectively carried out over the regions in which $p \sim p_0$, $\epsilon_{\mathrm{p}} \sim \omega_0$. We shall be interested in the values $q \ll p_0$, $\omega \ll \omega_0$. Therefore, these components contribute a real constant quantity to Π. The remaining component has the form

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*th = tanh.

$$\Pi'(\mathbf{q},\ i\omega_n) = \frac{1}{2(2\pi)^3}\int d^3\,\mathbf{p}g^2(\mathbf{p})\left\{(f'-f)\left(u^2u'^2+v^2v'^2-\frac{\Delta\Delta'}{2\varepsilon\varepsilon'}\right)\right.$$
$$\times\left(\frac{1}{\varepsilon-\varepsilon'-i\omega_n}+\frac{1}{\varepsilon-\varepsilon'+i\omega_n}\right)+\left(\operatorname{th}\frac{\beta\varepsilon}{2}+\operatorname{th}\frac{\beta\varepsilon'}{2}\right)$$
$$\left.\times\left(\frac{u^2v'^2+\Delta\Delta'/4\varepsilon\varepsilon'}{\varepsilon+\varepsilon'-i\omega_n}+\frac{v^2u'^2+\Delta\Delta'/4\varepsilon\varepsilon'}{\varepsilon+\varepsilon'+i\omega_n}\right)\right\}. \tag{27}$$

Here the unprimed quantities have the argument $\mathbf{p}$ and the primed ones the argument $\mathbf{p}-\mathbf{q}$. In order to go from the function (27), which is defined at the discrete values $\omega = i\omega_n$, to the function $\Pi'(\mathbf{q},\omega)$, which is analytic in the upper half-plane of ω, it is necessary to replace $i\omega_n$ by $\omega + i\delta$ everywhere in (14). Then we find $\operatorname{Re}\Pi'$, $\operatorname{Im}\Pi'$ by assuming ω to be real. For the reason given in the study of (26) we can regard $\operatorname{Re}\Pi'$ as a constant. $\operatorname{Im}\Pi$ has the form

$$\operatorname{Im}\Pi=\frac{\pi}{2(2\pi)^3}\int d^3\mathbf{p}\,g^2\left\{(f'-f)\left(u^2u'^2+v^2v'^2-\frac{\Delta\Delta'}{2\varepsilon\varepsilon'}\right)(\delta(\varepsilon-\varepsilon'-\omega)\right.$$
$$-\delta(\varepsilon-\varepsilon'+\omega))+\left(\operatorname{th}\frac{\beta\varepsilon}{2}+\operatorname{th}\frac{\beta\varepsilon'}{2}\right)\left[\left(u^2v'^2+\frac{\Delta\Delta'}{4\varepsilon\varepsilon'}\right)\right.$$
$$\left.\left.\times\,\delta(\varepsilon+\varepsilon'-\omega)-\left(u'^2v^2+\frac{\Delta\Delta'}{4\varepsilon\varepsilon'}\right)\delta(\varepsilon+\varepsilon'+\omega)\right]\right\}. \tag{28}$$

The components in (28) which are proportional to $\delta(\epsilon+\epsilon'\pm\omega)$ are brought about by processes of decay of the phonon into an electron-hole pair, and only make a non-zero contribution to the damping of the phonons above the threshold[11] $\omega_{\mathrm{th}} = 2\Delta$ ($\omega_{\mathrm{th}} \sim 10^{11}-10^{12}$ cps). We shall be interested in much lower frequencies for which the damping is determined by components with $\delta(\epsilon-\epsilon'\pm\omega)$.

We first note that both components are equal. Further, under our assumptions, we set $\Delta=\Delta'$, $u = u'$, $v = v'$, and $\epsilon = \epsilon'$ [of course, this is impossible to do in arguments of δ functions and in $f(\beta\epsilon')$]. Equation (28) then takes the form

$$\operatorname{Im}\Pi=\frac{1}{(2\pi)^2}\int d^3\,\mathbf{p}\,g^2(\mathbf{p})\,\frac{\xi^2}{\varepsilon^2}\,[f(\beta\varepsilon-\beta\omega)$$
$$-f(\beta\varepsilon)]\;\delta(\varepsilon-\varepsilon'-\omega). \tag{29}$$

The functions $f(\beta\epsilon)$, $f(\beta\epsilon-\beta\omega)$ limit the integration over ξ in (29) to the region $\xi\sim\Delta$. We then make the additional assumption that $vq \ll \Delta$ (for real phonons, this corresponds to frequencies $\omega < 10^{10}$). Then, in the important region of integration, $\epsilon \gg vq$, and $\epsilon-\epsilon'$ can be put in the form

$$\varepsilon-\varepsilon'=\mathbf{vq}\,\xi/\varepsilon. \tag{30}$$

Substituting (30) in (29), and neglecting the small quantities $\sim c/v$, we get

$$\operatorname{Im}\Pi=\frac{1}{(2\pi)^2}\int d^3p\,\frac{\xi}{\varepsilon}\,(f(\beta\varepsilon-\beta\omega)-f(\beta\varepsilon))\,g^2\delta(\mathbf{vq}). \tag{31}$$

We transform in the integral (31) to integration over ξ and over the surface $\xi = \mathrm{const}$. Making use of the fact that the integration over ξ is limited to the region $\xi = \mathrm{const}$, which is small in comparison with μ, we can transform the integral (31) in the following way:

$$\operatorname{Im}\Pi=\frac{2}{(2\pi)^2q}\int\frac{d\sigma}{v_F}\,g^2\,(\mathbf{n})\,\delta(\cos\chi)\int_{\Delta}^{\infty}d\varepsilon\,(f(\beta\varepsilon-\beta\omega)-f(\beta\varepsilon)), \tag{32}$$

where χ is the angle between the direction of the Fermi velocity and the direction of propagation of the sound. If $\omega \ll T$ (this case exists in real experiments), then we obtain

$$\operatorname{Im}\Pi=\frac{2}{(2\pi)^2}\,\frac{\omega}{q}\int\frac{d\sigma}{v_F}\,f(\beta\Delta)\,g^2(\mathbf{n})\,\delta(\cos\chi). \tag{33}$$

We note that for real phonons ($\omega\sim cq$) the ratio $\operatorname{Im}\Pi/\operatorname{Re}\Pi$ is of the order $(c/v)e^{-\beta\Delta}$, where we choose for Δ the minimum value of $\Delta(\mathbf{p})$ on the line $\cos\chi = 0$.

Equation (14), which determines the ultrasonic dispersion, has the form

$$\frac{\pi^2}{p_0}\sum_s\frac{\lambda_s(\mathbf{n})\,\omega_s^{02}(\mathbf{q})}{\omega_s^{02}(\mathbf{q})-\omega^2}=\frac{1}{\operatorname{Re}\Pi}-i\,\frac{\operatorname{Im}\Pi}{(\operatorname{Re}\Pi)^2}. \tag{34}$$

The first term on the right side of (34) reduces to the finite renormalization of the sound velocity, while the second corresponds to the damping. In view of its smallness, it is evident that the damping can be written in the form

$$\alpha_s(\mathbf{q})=\operatorname{Im}\omega_s(\mathbf{q})=\nu_s(\mathbf{n})\,\omega_s(\mathbf{q})\operatorname{Im}\Pi, \tag{35}$$

where $\nu_s(\mathbf{n})$ is some function of the direction that has not been measured experimentally.

Equations (35) and (33) show that the damping of the sound has a clearly expressed anisotropy.

We can eliminate the function $\nu_s(\mathbf{n})$ by considering the ratio of the damping of the sound in the superconductor (35) to the corresponding value $\alpha_n(\mathbf{q})$ in the normal metal, since the formula for $\alpha_n(\mathbf{q})$ is identical with (35) if we set $f(\beta\Delta) = \frac{1}{2}$ in Eq. (33) for $\operatorname{Im}\Pi$. The formula obtained by elimination of $\nu_s(\mathbf{n})$ is the analogue of Eq. (1) for the anisotropic case:

$$\frac{\alpha_s}{\alpha_n}=2\,\langle f(\beta\Delta)\rangle_{\mathbf{n}},$$
$$\langle\varphi\rangle_{\mathbf{n}}=\int\frac{d\sigma'}{v_F'}\,g^2(\mathbf{n}')\,\delta(\cos\chi)\,\varphi(\mathbf{n}')\bigg/\int\frac{d\sigma'}{v_F'}\,g^2(\mathbf{n}')\,\delta(\cos\chi). \tag{36}$$

Unfortunately, a second unmeasurable function $g^2(\mathbf{n})$, which enters into (36), does not permit us to draw any conclusions about $\Delta(\mathbf{n})$.

4. ULTRASONIC ABSORPTION IN THE REGION OF LOW TEMPERATURES

Let us now consider the region of low temperatures $\beta\Delta \ll 1$. We can carry out integration in Eq. (29) over ξ, obtaining

$$\operatorname{Im}\Pi = \frac{\beta\omega c^2}{(2\pi)^2 q}\int\frac{d\sigma}{v_F^4}\frac{g^2\Delta}{|\cos^3\chi|}\exp\left[-\beta\Delta\left(1-\frac{c^2}{v_F^2\cos^2\chi}\right)^{-1/2}\right]. \tag{37}$$

Here the integration is not extended over the region $\cos\chi < c/v_F$. We transform to integration over the stereographic projection of the Fermi surface, where we use χ as the polar angle:

$$d\sigma = K^{-1}(\chi, \varphi)\, d(\cos\chi)\, d\varphi, \tag{38}$$

where $K(\chi, \varphi)$ is the Gaussian curvature of the Fermi surface. Let (χ_0, φ_0) be the point at which $\Delta(\chi, \varphi)$ has an absolute minimum, equal to Δ_0, and $\cos\chi_0 \sim 1$. Then (χ_0, φ_0) is a stationary point of the integrand of (37), the contribution of which I_0 in $\operatorname{Im}\Pi$ is determined by the equation

$$I_0 = \frac{g^2}{4\pi\rho}\frac{\omega}{q}\frac{c^2}{Kv_F^4}\frac{1}{|\cos^3\chi_0|}\exp\left[-\beta\Delta_0\left(1-\frac{c^2}{v_F^2\cos^2\chi}\right)^{-1/2}\right],$$

$$\rho\Delta_0 = \frac{\partial^2\Delta}{\partial(\cos\chi)^2}\frac{\partial^2\Delta}{\partial\varphi^2} - \left(\frac{\partial^2\Delta}{\partial(\cos\chi)\,\partial\varphi}\right)^2. \tag{39}$$

In Eq. (39), all the functions are taken at the point of the minimum (χ_0, φ_0).

Because of the smallness of c/v_F the region of small $\cos\chi$ can make a relatively large contribution to the integral (37). For the investigation of the contribution of I_1 in $\operatorname{Im}\Pi$ from this region, it is convenient to introduce the variable $u = c/v_F \times \cos\chi$. We have

$$I_1 = \frac{\beta\omega}{4\pi^2 q}\int_0^{2\pi}d\varphi\int_{c/v_F}^{1}\frac{du}{Kv_F^2}\, g^2\Delta\exp[-\beta\Delta(1-u^2)^{-1/2} + \ln u]. \tag{40}$$

The equation determining the saddle point has the form

$$\frac{\beta\Delta u}{(1-u^2)^{3/2}} - \frac{1}{u} - \beta\frac{\partial\Delta}{\partial(\cos\chi)}\frac{c}{v_F u^2}(1-u^2)^{-1/2} = 0. \tag{41}$$

We shall consider the last component on the right side of (41) to be small (below, we shall explain when this is valid). Then Eq. (41) has the solution $u = (\beta\Delta)^{-1/2}$. The condition under which the approximations we have made are valid has the form $(\beta\Delta)^{3/2} c/v_F \ll 1$. In this approximation, I_1 reduces to the form

$$I_1 = \sqrt{\frac{\pi}{2e}}\frac{\omega}{4\pi^2 q}\int_0^{2\pi}d\varphi\,\frac{g^2}{Kv_F^2}e^{-\beta\Delta}. \tag{42}$$

Here all the functions are evaluated for $\cos\chi = 0$. The integration over φ in (42) can also be carried out in explicit form by making use of the large quantity $\beta\Delta$:

$$I_1 = \frac{1}{4\pi\sqrt{2e\sigma}}\frac{\omega}{q}\frac{g^2}{Kv_F^2}(\beta\Delta_0)^{-1/2}\exp(-\beta\Delta^{(\mathbf{n})}), \tag{43}$$

where $\Delta^{(\mathbf{n})}$ is the minimum value of Δ on the circle $\cos\chi = 0$ perpendicular to the direction $\mathbf{n}$; $\sigma\Delta_0$ is the value of the derivative $\partial^2\Delta/\partial\varphi^2$ at the minimum point on the circle. From (39) and (43), we get

$$I_1/I_0 \approx (v^2/c^2\sqrt{\beta\Delta})\exp\{-\beta(\Delta^{(\mathbf{n})} - \Delta_0)\}. \tag{44}$$

Let us consider two limiting cases.

1. Let $I_1/I_0 \gg 1$ in (44). For the majority of "pure" superconductors, only this case is amenable to experimental investigation at the present time (since the attainable values of $\beta\Delta$ have the order of ~ 10). We note that the condition $(\beta\Delta)^{3/2} c/v_F \ll 1$ is presumed to be satisfied beforehand.

With the aid of (35) and (43) we obtain

$$\alpha_s \sim \omega_s(\mathbf{q})\sqrt{T/T_c}\exp(-\Delta^{(\mathbf{n})}/T). \tag{45}$$

Equation (45) shows that α_s increases exponentially with temperature, while the exponent depends on the direction. This conclusion is confirmed by experimental data given in references 5 and 6.

We proceed to the problem of establishing $\Delta(\mathbf{n})$ from the known function $\alpha_s(\mathbf{q})$. For simplicity, we consider the case of azimuthal symmetry of $\Delta(\mathbf{n})$. Let the direction corresponding to the minimum of $\Delta(\mathbf{n})$ coincide with the axis of symmetry, and $\Delta(\vartheta)$ increase monotonically from the pole to the equator. Then the following formula clearly holds:

$$\Delta(\vartheta) = \Delta(\pi/2 - \vartheta). \tag{46}$$

Thus in this case the experimental measurements make it possible to establish $\Delta(\vartheta)$ completely and uniquely.

However, in the case in which some finite angle ϑ_0 corresponds to the minimum of $\Delta(\vartheta)$ on the sphere, the values of $\Delta(\vartheta)$ [and, consequently, $\ln\alpha_s(\vartheta)$] for the angles $\vartheta > \pi/2 - \vartheta_0$ will be identical and Eq. (46) makes it possible to establish $\Delta(\vartheta)$ only for the angles $\vartheta > \vartheta_0$ (the symmetry $\vartheta \rightleftarrows \pi - \vartheta$ is presumed everywhere). In particular, for $\vartheta_0 = \pi/2$, we can only establish the values of the minimum of Δ_0.

The situation considered above is obviously typical in the sense that in the establishment of $\Delta(\mathbf{n})$ according to $\alpha_s(\mathbf{q})$ "white spots" appear on the sphere. Detailed analysis of the different possibilities which arise in the absence of azimuthal symmetry, the presence of certain minima of $\Delta(\mathbf{n})$, etc., will be given in a separate communication.

2. Let $I_1/I_0 \ll 1$ in (44). Making use of (35) and (43), we get

$$\alpha_s(\mathbf{q}) = A\nu_s(\mathbf{n})\,\omega_s(\mathbf{q})\,(c/v_F)^2 e^{-\beta\Delta_0}\sum|\cos^3\chi_0|^{-1}, \tag{47}$$

where χ is the angle between the direction $\mathbf{q}$ and the velocity at the minimum point of Δ, A is a constant of the order of unity (the summation is extended over all points of absolute minimum of Δ). In the derivation of Eq. (47), it was assumed that the coincidence of the values of $\Delta(\mathbf{n})$ and the different points of the minimum is not accidental but a consequence of the symmetry.

The behavior of α_s predicted by Eq. (47) is substantially different from the first case. We turn our attention to the anomalous behavior of α_s according to the directions corresponding to the small $\cos\chi$.*

It is interesting that for Nb, the transition temperature of which is about 8° K, the region for the second case is perhaps experimentally attainable at the present time and Eq. (47) can be checked experimentally.

Up to now we have assumed that the stereographic projection of the Fermi surface is an entire sphere. However, for open Fermi surfaces the stereographic projection is only part of a sphere. For surfaces of the "corrugated" type (see reference 14), the unoccupied part of the sphere can include large regions. If the sound is propagated in a direction perpendicular to the "unoccupied" region, then $\cos\chi$ does not have small values on the Fermi surface and consequently the ultrasonic attenuation is described by Eq. (47).

We emphasize that in the given range of directions the ultrasonic attenuation is described by Eq. (47) under the condition $\beta\Delta \gg 1$, i.e., for temperatures much higher than required by the condition of the second case.

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Translated by R. T. Beyer
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*Of course, $\cos\chi$ may not be set equal to zero, since the condition $|\cos\chi| \geq c/v_F$ must be fulfilled.

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GENERALIZATION OF THE KRAMERS-KRONIG FORMULAS TO MEDIA WITH SPATIAL DISPERSION

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A condition that must be satisfied by the complex electric permittivity of a medium with spatial dispersion, and that generalizes the Kramers-Kronig formulas, is derived by requiring satisfaction of the causality principle in a form following from the theory of relativity. The conditions obtained, in contrast to the Kramers-Kronig formulas, not only relate the real and imaginary parts of the permittivity to each other but also impose restrictions on each of them separately.

As is known, the complex dielectric permittivity $\epsilon(\omega)$ of a medium, as a function of ω, must satisfy the integral equation

$$\varepsilon(\omega) - 1 = \frac{1}{i\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon(\zeta) - 1}{\zeta - \omega} d\zeta + \frac{4\pi\sigma_0 i}{\omega} \tag{1}$$

(here P indicates that the integral at the point $\zeta = \omega$ is to be interpreted as the principal value; σ_0 is the conductivity at $\omega = 0$). This equation is equivalent to equations for the real and imaginary parts of $\epsilon(\omega)$ that are known as the Kramers-Kronig formulas (cf., for example, reference 1, Sec. 62). If instead of the complex permittivity we use the complex conductivity $\sigma(\omega) = -i\omega \times (\epsilon(\omega) - 1)/4\pi$, then instead of (1) we get for $\sigma(\omega)$ the equation

$$\sigma(\omega) = \frac{1}{i\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\sigma(\zeta)}{\zeta - \omega} d\zeta, \tag{2}$$

which can also be written in the form

$$\sigma(\omega) = \frac{1}{i\pi} \int_{0}^{\infty} \{\sigma(\omega + \eta) - \sigma(\omega - \eta)\} \frac{d\eta}{\eta}, \tag{2'}$$

equivalent to (2). On setting $\sigma = \sigma' - \sigma''$, we get from (2) the Kramers-Kronig formulas

$$\sigma'(\omega) = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\sigma''(\zeta)}{\zeta - \omega} d\zeta, \qquad \sigma''(\zeta) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\sigma'(\zeta)}{\zeta - \omega} d\zeta. \tag{3}$$

One of the functions, $\sigma'(\omega)$ or $\sigma''(\omega)$, may be given; then the other is determined by (3). It must be remembered that either of the equations (3) implies the other as a consequence.

These formulas are obtained from a condition imposed by the principle of causality. Namely, since the field is the cause of the current, the linear integral operator that connects the total current $j(t)$ with the field $E(t)$ must have the form

$$j(t) = \int_{0}^{\infty} S(\tau) E(t - \tau) d\tau,$$

where the integration extends only over instants of time $t - \tau$ preceding the instant t. Accordingly, the complex conductivity $\sigma(\omega)$, which relates the current and the field $[j = \sigma(\omega) E]$ when the field varies according to the factor $e^{-i\omega t}$, is expressed by the one-sided Fourier integral (between limits 0 and ∞)

$$\sigma(\omega) = \int_{0}^{\infty} S(\tau) e^{i\omega\tau} d\tau. \tag{4}$$

Equation (2) or (2′) is the condition that the Fourier transform $s(\tau)$ of the function $\sigma(\omega)$ vanish for negative τ, so that the Fourier expansion of $\sigma(\omega)$ may be a one-sided integral (cf., for example, reference 2, Secs. 5.1 — 5.4).

In a medium with spatial dispersion, the current at a given point is determined by the field not just at that point but at neighboring points as well. The relation between the current and the field in this case is expressed by an integral both over time and over space. In this case the requirement imposed by the causality principle, with attention to the requirements of the theory of relativity, leads to the result that in this integral the integration must be extended over the region inside a cone of light facing the past. For our purpose it is sufficient to consider the case in which the field depends only on the coordinate x. Then for a homogeneous medium the requirement mentioned clearly leads to the following expression relating the current to the field:

$$j(t,x) = \int_{0}^{\infty} d\tau \int_{-\tau}^{\tau} S(\tau,\xi) E(t - \tau, x - \xi) d\xi .$$

Here the unit of time has been so chosen that $c = 1$. On setting $E(t, x) = \exp[-i(\omega t + kx)]$, we get $j = \sigma(\omega, k) E$, with the complex conductivity

$$\sigma(\omega,k) = \int_0^\infty e^{i\omega\tau} d\tau \int_{-\tau}^{\tau} S(\tau, \xi)\, e^{ik\xi}\, d\xi\,. \qquad (5)$$

From the fulfillment of this necessary condition of relativity theory it follows, in particular, that the leading wave front is propagated with the speed of light. This conclusion is quite natural, since now the equations of electrodynamics (including even the equation that relates the current to the field) satisfy the requirements of relativity theory. An analytic proof of this conclusion is also quite simple. (This conclusion, obviously, is correct also for media without spatial dispersion, when the causality requirement is fulfilled in the form indicated above; and it implies that in the limit $\sigma(\omega) \to 0$ as $\omega \to \infty$.)

As is known,[3,4] the complex conductivity of isotropic media, which we are here considering, is described by a tensor of the form

$$\sigma_{ij} = \left(\delta_{ij} - \frac{k_i\, k_j}{k^2}\right) \sigma^t(\omega,k) + \frac{k_i k_j}{k^2}\, \sigma^l(\omega,k)\,.$$

Although we have not explicitly taken account of this circumstance, we obviously may consider separately transverse and longitudinal waves; thus the quantity $\sigma(\omega, k)$ may be either the transverse conductivity σ^t or the longitudinal one σ^l. The conductivity $\sigma(\omega, k)$ depends only on the absolute value k of the vector $\mathbf{k}$.* Furthermore, since the function $S(\tau, \xi)$ is real, the real part of $\sigma(\omega, k)$ is an even function of ω, and the imaginary part is an odd function.

It is obvious that for fixed k, the conductivity $\sigma(\omega, k)$, Eq. (5), as a function of ω can be expressed as a semi-infinite Fourier integral over τ, and reduces to the form (4) if we set

$$S(\tau) = \int_{-\tau}^{\tau} S(\tau, \xi)\, e^{ik\xi}\, d\xi\,.$$

Therefore $\sigma(\omega, k)$ as a function of ω must (for arbitrary fixed k) satisfy Eq. (2′), i.e.,

$$\sigma(\omega, k) = \frac{1}{i\pi} \int_0^\infty \{\sigma(\omega + \eta, k) - \sigma(\omega - \eta, k)\} \frac{d\eta}{\eta}. \qquad (6)$$

We may, however, write a more general equation for $\sigma(\omega, k)$ if we notice that Eq. (5) preserves its form when the variables τ and ξ undergo a Lorentz transformation (with transformation of the scale of both variables). In fact, under this transformation the light cone is transformed into itself. At the same time σ goes over into a function of variables ω' and k', related to ω and k also by a Lorentz transformation (inverse to the transformation of τ and ξ):

$$\omega = \gamma(\omega' - \beta k'), \quad k = \gamma(k' - \beta\omega'), \quad |\beta| \leqslant 1. \qquad (7)$$

From this it follows that $\sigma(\gamma(\omega' - \beta k'), \gamma(k' - \beta\omega'))$, as a function of ω' (for fixed k'), will also satisfy condition (6), i.e.,

$$\sigma(\gamma(\omega' - \beta k'), \gamma(k' - \beta\omega'))$$
$$= \frac{1}{i\pi} \int_0^\infty \{\sigma(\gamma(\omega' + \eta' - \beta k'), \gamma(k' - \beta\omega' - \beta\eta'))$$
$$- \sigma(\gamma(\omega' - \eta' - \beta k'), \gamma(k' - \beta\omega' + \beta\eta'))\} \frac{d\eta'}{\eta'}.$$

By transforming now to the variables ω and k and setting $\eta = \gamma\eta'$, we get

$$\sigma(\omega, k) = \frac{1}{i\pi} \int_0^\infty \{\sigma(\omega + \eta, k - \beta\eta) - \sigma(\omega - \eta, k + \beta\eta)\} \frac{d\eta}{\eta}, \qquad (8)$$

where β has an arbitrary value in the interval $|\beta| \le 1$.

We shall now show that in order that $\sigma(\omega, k)$ may be expressible in the form of a Fourier integral (5) extended over the quadrant $|\xi| < \tau$, it is sufficient that condition (8) be satisfied for $\beta = +1$ and for $\beta = -1$. For this purpose we find the Fourier transform of a function $\sigma(\omega, k)$ satisfying condition (8).

This will be

$$S(\tau, \xi) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\omega\tau + k\xi)}\, \sigma(\omega, k)\, d\omega\, dk.$$

Here τ and ξ may take all values from $-\infty$ to $+\infty$. On inserting $\sigma(\omega, k)$ from (8), changing the order of integration, and making the transformation of variables $\omega \pm \eta = \omega_1$, $k \mp \beta\eta = k_1$, we get

$$S(\tau, \xi) = \frac{1}{2\pi^8} \int_0^\infty \frac{\sin(\tau - \beta\xi)\,\eta}{\eta}\, d\eta$$
$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\omega_1\tau + k_1\xi)}\, \sigma(\omega_1, k_1)\, d\omega_1\, dk_1.$$

By noting that

$$\int_0^\infty \sin\alpha\eta\, \frac{d\eta}{\eta} = \frac{\pi}{2} \operatorname{sgn}\alpha,$$

we get

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*It is not difficult to show that the possibility of expressing the function $\sigma(\omega, k)$ in the form (5) is equivalent to the possibility of expressing it in the form of an integral over space:

$$\sigma(\omega, k) = \int_0^\infty e^{i\omega\tau} d\tau \int_{r<\tau} Q(\tau, r)\, e^{i\mathbf{kr}}\, d\mathbf{r}.$$

where

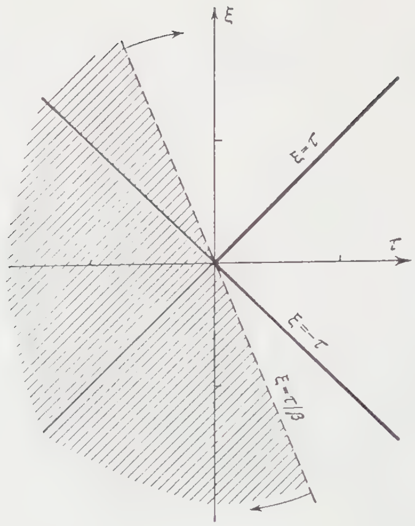
$$O(\tau, r) = (1/2\pi r)\partial S(\tau, r)/\partial r.$$

$$S(\tau, \xi) = S(\tau, \xi)\,\mathrm{sgn}\,(\tau - \beta\xi)$$

or

$$S(\tau, \xi)\,[1 - \mathrm{sgn}\,(\tau - \beta\xi)] = 0. \tag{9}$$

Hence it is clear that $S(\tau, \xi)$ can be different from zero only where $\tau - \beta\xi$ is positive.



This means that for fixed β the function $S(\tau, \xi)$ can be different from zero to the right of the straight line $\xi = \tau/\beta$ (in the unshaded half-plane in the figure). When β varies from -1 to $+1$, this straight line rotates clockwise, taking all positions between the straight lines $\xi = -\tau$ and $\xi = \tau$ (which bound the quadrant of interest to us, $|\xi| < \tau$). It is clear that from condition (9) merely for $\beta = -1$ and $\beta = +1$ it follows that $S(\tau, \xi)$ is different from zero only in this quadrant; and this means that the Fourier expansion of $\sigma(\omega, k)$ has the form (5). Satisfaction of condition (8) for $\beta = +1$ and for $\beta = -1$ is thus sufficient. Since (5) follows from this, satisfaction of condition (8) for other values of β ($|\beta| \leq 1$) is a consequence of satisfaction of it for $\beta = +1$ and for $\beta = -1$.

The complex conductivity $\sigma(\omega, k)$, which is of interest to us, is an even function of k. It is easy to show that for an even function, the conditions (8) written for $\beta = +1$ and for $\beta = -1$ coincide. Therefore our function $\sigma(\omega, k)$ needs to satisfy a single condition, which is obtained from (8) by setting $\beta = 1$. If we now go over to the usual units of time, k must be replaced by kc; then on transforming (8) to a form analogous to (2), we get

$$\sigma(\omega, k) = \frac{1}{i\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\sigma(\zeta, k + \beta(\zeta - \omega)/c)}{\zeta - \omega}\, d\zeta. \tag{10}$$

It is sufficient that $\sigma(\omega, k)$ satisfy this condition for $\beta = 1$; then $\sigma(\omega, k)$ will satisfy it for arbitrary β ($|\beta| \leq 1$).

Equation (10) gives the generalization of the Kramers-Kronig formulas for a medium with spatial dispersion. As $c \to \infty$, (10) becomes usual Kramers-Kronig equation (2).

It is necessary, however, to pay attention to the following important difference in the conditions imposed on $\sigma(\omega, k)$. In the usual case, without spatial dispersion, one of the functions, $\sigma'(\omega)$ or $\sigma''(\omega)$ ($\sigma = \sigma' - i\sigma''$), may be specified arbitrarily, whereas the other is determined by a Kramers-Kronig formula. The two formulas (3), relating $\sigma'(\omega)$ and $\sigma''(\omega)$, are not independent, one being a consequence of the other (for example, substitution in the first formula of $\sigma''(\omega)$ from the second leads to an equality containing only $\sigma'(\omega)$ and satisfied identically). Whereas $\sigma(\omega)$ is expressed in the form of the semi-infinite Fourier integral (4), $\sigma'(\omega)$ and $\sigma''(\omega)$ are expressed in the form of the general Fourier integral, extended from $-\infty$ to $+\infty$, but their Fourier transforms can be expressed in terms of each other in a known fashion:

$$\sigma'(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} S(\tau)\, e^{i\omega\tau}\, d\tau,$$

$$\sigma''(\omega) = \frac{1}{2i} \int_{-\infty}^{\infty} S(\tau)\, e^{i\omega\tau}\, \mathrm{sgn}\,\tau \cdot d\tau,$$

$$S'(-\tau) = S(\tau).$$

In contrast to this, condition (10) not only relates the real and imaginary parts of $\sigma(\omega, k)$, but also imposes restrictions on each of them separately. For since [by (5)] $\sigma(\omega, k)$ is expressed by a Fourier integral extended only over the single quadrant $|\xi| < \tau$ therefore $\sigma'(\omega, k)$ and $\sigma''(\omega, k)$ will be expressed by Fourier integrals extended only over two quadrants ($|\xi| < \tau$ and the one opposite it), and not over the whole (ξ, τ) plane. This imposes a condition on each of the functions $\sigma'(\omega, k)$ and $\sigma''(\omega, k)$.

It is still necessary to add the following. Whereas the longitudinal conductivity $\sigma^l(\omega, k)$ must always satisfy condition (10), the transverse conductivity $\sigma^t(\omega, k)$ satisfies this condition only for a nonmagnetic medium; more accurately, for a medium whose static magnetic permeability is unity. In the general case we have[3,4]

$$\sigma^t(\omega, k) = \sigma^l(\omega, k) - \frac{ik^2c^2}{\omega}\left(1 - \frac{1}{\mu(\omega, k)}\right), \tag{11}$$

where $\mu(\omega, k)$ is the magnetic permeability, dependent on ω and k. Whereas $\sigma^l(\omega, k)$, describing the electrical properties of the medium, has no singularity at $\omega = 0$, the function $\sigma^t(\omega, k)$, as is clear from (11), has a singular point (pole) there if $\mu(0, k) \neq 1$.

To generalize Eq. (10) to this case, it is necessary to proceed in the following manner.

The quantity $\partial j/\partial t$ will be related to the field E by an integral operator of the same type as that relating j to E; and, clearly, this operator will also satisfy the requirements of relativistic causality. For a monochromatic plane transverse wave

$$\partial j/\partial t = -i\omega\sigma^t(\omega, k)\,E.$$

Consequently $\omega\sigma^t(\omega, k)$ must satisfy an equation of the form (10) (since $\omega\sigma^t$ no longer has a singular point at $\omega = 0$), i.e., the equation

$$\omega\sigma^t(\omega, k) = \frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\zeta\sigma^t(\zeta, k+\beta(\zeta-\omega)/c)}{\zeta-\omega}\,d\zeta. \tag{12}$$

Hence after some transformations we get

$$\sigma^t(\omega, k) = \frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\sigma^t(\zeta, k+\beta(\zeta-\omega)/c)}{\zeta-\omega}\,d\zeta$$

$$+\frac{1}{i\pi\omega}\,\mathrm{P}\int_{-\infty}^{\infty}\sigma^t\left(\zeta, k+\frac{\beta}{c}(\zeta-\omega)\right)d\zeta. \tag{13}$$

Here the integrals are to be interpreted as principal values at the points $\zeta = 0$ and $\zeta = \omega$. From (12) we get

$$\frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\sigma^t\left(\zeta, k+\frac{\beta}{c}(\zeta-\omega)\right)d\zeta = \lim_{\omega'=0}\omega'\,\sigma^t\left(\omega', k-\frac{\beta}{c}\omega\right),$$

and from (11) we find

$$\lim_{\omega'=0}\omega'\sigma^t\left(\omega', k-\frac{\beta}{c}\omega\right)$$

$$= -i\left(k-\frac{\beta}{c}\omega\right)^2 c^2\left(1-\frac{1}{\mu(0, k-\beta\omega, c)}\right).$$

On substituting this value in (13), we get

$$\sigma^t(\omega, k) = \frac{1}{i\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\sigma^t(\zeta, k+\beta(\zeta-\omega)/c)}{\zeta-\omega}\,d\zeta$$

$$-\frac{i(kc-\beta\omega)^2}{\omega}\left(1-\frac{1}{\mu(0, k-\beta\omega/c)}\right). \tag{14}$$

In closing, it should be indicated how the conclusions deduced above carry over to an anisotropic medium, with spatial dispersion described by a complex conductivity tensor $\sigma_{ij}(\omega, \mathbf{k})$ whose components depend both on the absolute value of k and on the direction of the vector **k**. It is easy to show that the requirements of relativistic causality impose restrictions not on the dependence of σ_{ij} on the direction of **k**, but only on its dependence on the absolute value k. Thus in this case each component σ_{ij}, as a function of ω and k, satisfies the previous equation (10) [or, in the general case, (14)].

I thank V. P. Silin and N. N. Meĭman for fruitful discussion.

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X-RAY SCATTERING IN CRYSTALS WITH EXCITON FORMATION

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A study is made of Raman scattering of x rays in crystals accompanied by the formation of excitons, i.e., "normal" electromagnetic waves in the given crystal. The plasmons (longitudinal plasma waves) in metals and dielectrics are a special case of excitons. The measurement of the energy of the scattered x-ray photons at various scattering angles may be an effective method of determining the energy of the excitons as a function of the wave vector.

1. As is well known, the excitons occurring in optical studies are nothing but the "normal" electromagnetic waves propagated in the given medium.[1-4] Hence, the determination of the exciton energy $\hbar\omega(\mathbf{k})$ as a function of the wave vector $\mathbf{k}$ is equivalent to measuring the refractive index n of the crystal for waves of frequency ω propagated in the direction of $\mathbf{k}$.

However, for a number of reasons, it is difficult to work in optics with values of $n \gtrsim 10$, that is, with $k = 2\pi/\Lambda = 2\pi n/\lambda \gtrsim 10^6\ \text{cm}^{-1}$. (Here $\lambda = 2\pi c/\omega$ is the wavelength in vacuo; we assume for estimates in the optical region that $\lambda \sim 5 \times 10^{-5}$ cm; Λ is the wavelength in the given substance.) Correspondingly, it is difficult to determine one of the fundamental characteristics of excitons, the function $\omega(\mathbf{k})$, when $k \gtrsim 1-3 \times 10^6\ \text{cm}^{-1}$. In particular, no one has yet given by optical methods a proof at all convincing of the existence of a "new" type of waves in crystals in the vicinity of absorption lines.[5,6] (The "new" waves merely amount to the fact that, when we take into account the spatial dispersion at the given frequency ω in the given direction, we find that not two, but a larger number of "normal" electromagnetic waves can be propagated. For further detail, see references 6 and 7.)

In view of the above, the use of non-optical methods to determine the function $\omega(\mathbf{k})$ is of great interest. If the exciton has a character essentially related to the vibrations of ions, as is the case in ionic crystals for the so-called optical vibrations,* an effective method of investigation may be the study of inelastic neutron scattering.[9] We note that this method is especially effective in the study of acoustic lattice vibrations. However, in the case of excitons of the "electronic" type, weakly bound to the lattice vibrations, we may rely on other methods: the measurement of discrete energy losses during passage of electrons through thin films, and the study of Raman (inelastic) scattering of x rays in crystals. It is the latter phenomenon which we shall consider here.

2. Obviously, the topic here is the measurement of the frequency ω_0' of x rays scattered at an angle θ by a crystal, with generation of an exciton. (The frequency of the incident radiation is ω_0; θ is the angle between the wave vectors of the scattered and incident waves $\mathbf{k}_0'$ and $\mathbf{k}_0$.) From the conservation laws we have (neglecting the absorption of excitons, which will lead to a smearing of the pattern)

$$\omega = \omega_0 - \omega_0', \quad \mathbf{k} = \mathbf{k}_0 - \mathbf{k}_0',$$

$$2\Lambda \sin(\theta/2) \equiv (2\lambda/n(\omega)) \sin(\theta/2) = \lambda_0 = 2\pi c/\omega_0, \tag{1}$$

where, in deriving the latter expression, we have taken into account the fact that $n(\omega_0) \approx 1$, and $\omega \ll \omega_0$.*

Because of the crystal anisotropy, the frequency ω will depend, in general, for a given angle θ and crystal orientation, on the azimuth angle as well (i.e., on the direction of the vector $\mathbf{k}$ for the given θ). In addition, of course, the frequencies $\omega(\mathbf{k})$ differ for excitons of differing types, e.g., of different polarization. If we assume $\lambda_0 \sim 1-3$ A, and choose the lowest attainable angles ($\theta \sim 10'$),† then $\Lambda \sim (3-10) \times 10^{-6}$ cm; for $\theta \sim 1° - 3°$, we already find that $\Lambda \sim 2 \times 10^{-7} - 2 \times 10^{-6}$ cm, or $k = 2\pi n/\lambda \sim 3 \times 10^6 - 3 \times 10^7\ \text{cm}^{-1}$.

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*Of course, a terminology in which every type of electromagnetic wave in the medium (or photon in the medium) is called an exciton may sometimes be inconvenient. We shall, however, use it here. We shall also refer to the so-called plasmons, i. e., longitudinal electromagnetic, or more precisely, "electric" waves, as excitons. For more detail on the relation between the theory of excitons and crystal electrodynamics, see reference 8.

*With regard to radiation scattered in the vicinity of a Bragg maximum, see below.

†The authors are grateful to V. V. Shmidt for information on the experimental limitations.

Typical energy values for excitons lie within the limits $\hbar\omega \sim 0.1 - 10$ ev. We may point out for comparison that in scattering by free electrons $\hbar\omega \approx (\hbar\omega_0/2mc^2)\,\theta^2\hbar\omega_0$. Thus, if $\theta = 1°$ and $\lambda_0 \sim 3$ A, then the energy loss of the x-ray photon $\hbar\omega \sim 4 \times 10^{-3}$ ev. The resolving power of x-ray spectroscopic apparatus in energy terms is at most only ~ 0.3 ev (at $\lambda_0 \sim 3$ A); this limits considerably the possibilities of the method, but still, when $\hbar\omega \gtrsim 0.5 - 1$ ev, we may hope to obtain valuable information on the function $\omega(\mathbf{k})$, provided that the intensity of the Raman scattering is sufficient.*

3. The physical picture is especially clear in the limiting case in which the frequency of the x-rays is large in comparison with all characteristic frequencies of the medium. In this case, as is known (see, e.g., reference 11), the effective coherent-scattering cross-section for unpolarized radiation is given by the formula

$$d\sigma = r_0^2\,\frac{1+\cos^2\theta}{2}\left|\int N_0(\mathbf{r})\,e^{i\mathbf{kr}}\,d\mathbf{r}\right|^2 d\Omega, \tag{2}$$

where $N_0(\mathbf{r})$ is the time-average electron density, $r_0 = e^2/mc^2$, and $d\Omega = \sin\theta\,d\theta\,d\varphi$.

We may obtain this result by the quantum-mechanical approach by taking the expression $(e^2/2mc^2)\,N_0(\mathbf{r})\,\mathbf{A}^2(\mathbf{r})$ for the interaction energy in the Hamiltonian in this problem. Here, $\mathbf{A}$ is the vector potential of the radiation field.

In the study of Raman scattering with the participation of an electromagnetic wave in the medium (an exciton), we can no longer consider the concentration $N(\mathbf{r})$ to be fixed, and we write

$$N(\mathbf{r}) = N_0(\mathbf{r}) + N'(\mathbf{r});$$

$$4\pi e N'(\mathbf{r}) = \operatorname{div}\mathbf{E}_j = -i(\mathbf{kj})\,E_j^0 \exp\,[i(\omega_j t - \mathbf{kr})]. \tag{3}$$

Here $\mathbf{E}_j(\mathbf{r}) = \mathbf{j}E_j^0 \exp\,[i(\omega_j t - \mathbf{k}\cdot\mathbf{r})]$ is the electric field in the "normal" wave (exciton) of type j being studied (the unit vector $\mathbf{j}$ defines the polarization of the wave).† The energy of this wave can be represented in the form of the energy of an oscillator

$$a_j(\mathbf{k})\,\overset{*}{E}_j E_j V = (\overset{*}{p}_j p_j + \omega_j^2 \overset{*}{q}_j q_j) = 2\omega_j^2 \overset{*}{q}_j q_j = \hbar\omega_j l, \tag{4}$$

where the last result is obtained by quantization ($l = 0, 1, 2, \ldots$); V is the volume of the crystal, and the coefficient $a(\mathbf{k})$ can be calculated from a consideration of the energy of the system.

The energy of interaction with the x rays is now equal to [see Eqs. (3) and (4)]:

$$\frac{e^2}{2mc^2}\,[N_0(\mathbf{r}) + N'(\mathbf{r})]\,\mathbf{A}^2(\mathbf{r})$$

$$= \frac{e^2}{2mc^2}\left\{N_0(\mathbf{r}) - \frac{i(\mathbf{kj})\,q_j\omega_j\sqrt{2}}{4\pi e\sqrt{Va_j}}\,e^{i(\omega_j t - \mathbf{kr})}\right\}\mathbf{A}^2(\mathbf{r}). \tag{5}$$

If $l \gg 1$, the problem becomes classical, and is reduced to the scattering of transverse electromagnetic waves (x rays) by waves having a longitudinal field component (e.g., in an isotropic plasma only plasma waves are of interest, whereas in a magnetoactive plasma all waves are in general responsible for the scattering[7,12]). Crystals in an initial equilibrium state contain practically no excitons, and thus, we must consider the transition from the state $l = 0$ to the state $l = 1$ [see Eq. (4)]. The matrix element for such a transition is

$$(0\,|\,q_j e^{i\omega_j t}\,|\,1) = \sqrt{\hbar/2\omega_j}.$$

Here we need not solve the x-ray scattering problem anew, since by starting with Eq. (5) with $q_j = 0$, we can derive Eq. (2), while the calculation with $q_j \neq 0$ is essentially the same.

Thus by using Eqs. (2) and (5) together, we find the cross section for the Raman scattering:

$$d\sigma_k = r_0^2\,\frac{1+\cos^2\theta}{2}\,(\mathbf{kj})^2\,\frac{V\hbar\omega_j}{(4\pi e)^2 a_j}\,d\Omega$$

$$= r_0^2\,\frac{1+\cos^2\theta}{2}\,(\mathbf{kj})^2\,\frac{\hbar\omega_j \bar{N}_0 V}{4\pi a_j m\omega_p^2}\,d\Omega, \tag{6}$$

where $\omega_p^2 = (4\pi e^2/m)\,\bar{N}_0$ is the "plasma frequency," and $\bar{N}_0$ is the mean electron density.

For incoherent scattering by free electrons, we have in Eq. (2):

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*From the quantum viewpoint, the well-known thermal diffuse scattering of x-rays is Raman (inelastic, incoherent) scattering involving phonons. Raman scattering involving the lattice (or the ions) will not be considered below. For excitons having $\hbar\omega < 0.1$ ev, the frequency shift may be measured by means of the Mössbauer effect.[10]

†The "normal" waves used here, with $E_j^0 = \text{const}$, are the solutions of the field equations in which the dielectric-constant tensor $\varepsilon_{ij}(\omega, \mathbf{k})$ is introduced. For wavelengths comparable with the lattice parameter of the crystal, such a treatment is no longer accurate, since for the "normal" waves in the crystal, the amplitudes E_j^0 are periodic functions of $\mathbf{r}$ (with the period of the lattice). By writing the field in the "normal" wave in the form

$$\mathbf{E}_j = \mathbf{E}_{jk}^0(\mathbf{r})\exp\,[i(\omega_j t - \mathbf{kr})],$$

where $\mathbf{E}_{jk}^0(\mathbf{r})$ is a periodic function of $\mathbf{r}$ (with the lattice periodicity), we can calculate the Raman scattering in a way analogous to that used in the body of this article. The very important qualitative difference which arises here is the appearance of Raman scattering with $\mathbf{k}_0 - \mathbf{k}_0' = \mathbf{k} + \mathbf{b}$, where $\mathbf{b}$ is an integer reciprocal-lattice vector. The authors are grateful to L. V. Keldysh for a discussion of this question.

$$\left|\int N_0(\mathbf{r})\, e^{i\mathbf{kr}}\, d\mathbf{r}\right|^2 = \overline{N}_0 V,$$

since $N(\mathbf{r}) = \sum_k \delta(\mathbf{r} - \mathbf{r}_i)$. Hence, the ratio of the cross-section in Eq. (6) to the cross-section for scattering by free-electrons $d\sigma_0$ is equal to

$$\frac{d\sigma_k}{d\sigma_0} = \frac{(\mathbf{kj})^2\, \hbar\omega_j}{4\pi a_j\, m\omega_p^2} = \left(\frac{\hbar\omega_0}{mc^2}\right)\left(\frac{\omega_0\, \omega_j}{\omega_p^2}\right)\frac{\sin^2(\theta/2)\cos^2\psi}{\pi a_j}, \qquad (7)$$

where ψ is the angle between $\mathbf{k}$ and $\mathbf{j}$, and Eq. (1) has been used:

$$k^2 = (2\omega_0/c)^2 \sin^2(\theta/2).$$

In the propagation of waves along corresponding symmetry axes, or in any direction in cubic crystals (neglecting spatial dispersion[6]), the "normal" waves are divided into transverse and longitudinal waves. For transverse waves, $\mathbf{k}\cdot\mathbf{j} = 0$, and thus in the present approximation, no Raman scattering will arise [see Eqs. (6) — (7)].

Longitudinal waves (for which $d\sigma_k \neq 0$) are commonly called plasma waves (plasmons), since they are analogous to the longitudinal waves in an isotropic plasma. For plasma waves in a gas of free electrons, neglecting spatial dispersion (this means that $\omega_p^2 \gg v_0^2 k^2$, where v_0 is some mean electron velocity), $\omega_j = \omega_p$, and $a_j = \frac{1}{2}\pi$ (see, e.g., reference 7). However, in the case of any given optically-isotropic medium (metals, dielectrics) having sufficiently weak absorption in the given frequency range, we find for long longitudinal (plasma) waves

$$\varepsilon(\omega_j) = 0, \qquad a_j = (\omega_j/4\pi)\,(d\varepsilon/d\omega)\,\omega_j,$$

where $\epsilon(\omega)$ is the dielectric constant of the medium, and $d\epsilon/d\omega$ is evaluated for $\omega = \omega_j$ (for free electrons, $\epsilon = 1 - \omega_p^2/\omega^2$).

In the case of free electrons, we have

$$d\sigma_{k,p}/d\sigma_0 = \hbar k^2/2m\omega_p = 2\,(\hbar\omega_0/mc^2)\,(\omega_0/\omega_p)\,\sin^2(\theta/2). \qquad (8)$$

Equation (8) is equivalent to the corresponding formula derived in the classical calculation of scattering by a plasma wave $N' \exp[i(\omega_p t - \mathbf{k}\cdot\mathbf{r})]$ having an amplitude

$$|N'| = \sqrt{\hbar k^2 \overline{N}_0/2m\omega_p V}.$$

The wave amplitude has this value if we set $E^*EV/2\pi = \hbar\omega_p$.

If $\theta \sim 3°$, $\hbar\omega_0 \sim 5 \times 10^3$ ev ($\lambda_0 \sim 3$ A), and $\hbar\omega_p \sim 10$ ev, the ratio $d\sigma_{k,p}/d\sigma_0 \sim 10^{-2}$. Obviously, it follows that the intensity of the Raman scattering is sufficient for reliable measurement, provided that the actual values of the coefficients $1/a_j$ and $\cos^2\psi$ in Eq. (7) are not much smaller than the values used in deriving Eq. (8): $1/a_j = 2\pi$, and $\cos^2\psi = 1$.

4. In order to take into account the role of bound electrons and to specify the type of exciton considered, we shall study the model of a molecular crystal. In this case, the operator H' for the interaction with the radiation field can be written as a sum[4]

$$H' = H^{(1)} + H^{(2)}; \qquad (9)$$

$$H^{(1)} = -\frac{e}{mc}\sum_{\mathbf{n}\alpha,\, i_{\mathbf{n}\alpha}} \mathbf{A}(\mathbf{r}^i_{\mathbf{n}\alpha})\, \mathbf{p}^i_{\mathbf{n}\alpha}, \qquad (9a)$$

$$H^{(2)} = \frac{e^2}{2mc^2}\sum_{\mathbf{n}\alpha,\, i_{\mathbf{n}\alpha}} \mathbf{A}^2(\mathbf{r}^i_{\mathbf{n}\alpha}). \qquad (9b)$$

In Eqs. (9a) and (9b) $\mathbf{r}^i_{\mathbf{n}\alpha}$ is the radius vector of the i-th electron belonging to the $\mathbf{n}\alpha$-th molecule, $\mathbf{p}^i_{\mathbf{n}\alpha}$ is its momentum ($\mathbf{n}$ is an integral lattice vector, and α is the number of the molecule in the unit cell, $\alpha = 1, 2, \ldots \sigma$).

The operator $H^{(1)}$ gives the contribution to the intensity of Raman scattering of x-ray photons with formation of excitons only via transitions between intermediate states. It is easily seen that in calculating this contribution, the only appreciable interaction can be that of the x-ray photons with the strongly-bound electrons, whose binding energies are of the order of the x-ray photon energy.

The process of Raman scattering of x-ray quanta by individual atoms with production of plasmons has already been studied by Sobel'man and Feĭnberg,[13] who took into account the intermediate states of the K electrons.* It follows from the formula for the ratio $\sigma_{k,p}/\sigma_0$ given in reference 13 that intermediate states of the K electrons may contribute appreciably to the Raman-scattering cross section only in case that the energy of the x-ray photon is very near to the binding energy $|\epsilon_0|$ of the K electron. In this energy range, however, difficulties may arise in the observation of Raman scattering, owing to the nearness of the absorption edge. However, if $|\hbar\omega_0 - |\epsilon_0|| \sim \hbar\omega_0$ the contribution to the cross section calculated in reference 13 turns out to be small ($\sigma_{k,p}/\sigma_0 \lesssim 10^{-7}$) in comparison with the contribution due to direct transitions effected by the perturbation $H^{(2)}$. [Thus, it follows from Eq. (8) given above that $\sigma_{k,p}/\sigma_0 \sim 1$.] Hence, we shall limit ourselves below to calculating this fundamental contribution, in complete agreement with the calculations given in Sec. 3, having in mind an energy range of x ray quanta sufficiently distant from the absorption edge.

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*We note that reference 13 also considers the excitation of a plasmon upon absorption of an x-ray quantum. Of course, such a process may also involve the generation of an exciton of a more general type. However, in studies of x-ray absorption there is practically no opportunity to distinguish the excitons in terms of their momenta.

The vector potential of the x ray radiation field has the following form:[14]

$$\mathbf{A}(\mathbf{r}) = \sum_{qj} \sqrt{\frac{2\pi c^3\hbar}{Vqc}}\, \mathbf{l}_{qj} \left(a_{qj} e^{i\mathbf{qr}} + a^+_{qj} e^{+i\mathbf{qr}}\right), \tag{10}$$

where $\mathbf{l}_{qj}$ $(j = 1, 2)$ are unit vectors defining the transverse polarization of the photons $(\mathbf{q}j)$, and $a^+_{\mathbf{q}j}$ and $a^-_{\mathbf{q}j}$ are operators for the production and destruction of the photons $(\mathbf{q}j)$. Substitution of Eq. (10) into (9b) leads to a quadratic form with respect to the operators for production and destruction of photons. However, in this expression only the terms proportional to $a^-_{\mathbf{k}_0 j} a^+_{\mathbf{k}'_0 j'}$ give nonzero contributions to the cross section of the process being studied. If j and j' denote the polarization of the x-ray quantum before and after scattering, the perturbation operator which we must take into account is:

$$H^{(2)}(\mathbf{k}_0 j, \mathbf{k}'_0 j')$$
$$= \frac{2e^2\pi\hbar}{mcV} \sum_{\mathbf{n}\alpha} a_{\mathbf{k}_0 j}\, a^+_{\mathbf{k}'_0 j'} \frac{\mathbf{l}_{\mathbf{k}_0 j}\, \mathbf{l}_{\mathbf{k}'_0 j'}}{(k_0 k'_0)^{1/2}} \exp\{i(\mathbf{k}_0 - \mathbf{k}'_0,\ \mathbf{r}_{\mathbf{n}\alpha})\}\, F(\mathbf{n},\alpha). \tag{11}$$

The function

$$F(\mathbf{n}, \alpha) = \sum_{i_{\mathbf{n}\alpha}} \exp\{i(\mathbf{k}_0 - \mathbf{k}'_0,\ \mathbf{r}_i)\} \tag{12}$$

depends on the coordinates of the electrons of the molecule $(\mathbf{n}\alpha)$ with respect to the lattice point at which this molecule is situated.

In order to calculate the value of the cross section, we must know the wave function of the electrons of the crystal in the ground state as well as in the excited state accompanying the presence of the exciton. Neglecting retardation, which is not very substantial here,* these wave functions have the following form in the Heitler-London approximation (the weak effect of intermolecular exchange of electrons is not taken into account below):

$$\Phi_0 = \prod_{\mathbf{n}\alpha} \varphi^0_{\mathbf{n}\alpha}, \qquad \Phi_\mu(\mathbf{k}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{n}\alpha} B_{\alpha\mu}(\mathbf{k}) \exp(i\mathbf{k}\mathbf{r}_{\mathbf{n}\alpha})\, \Phi^f_{\mathbf{n}\alpha};$$
$$\Phi^f_{\mathbf{n}\alpha} = \varphi^f_{\mathbf{n}\alpha} \prod_{\mathbf{m}\beta \neq \mathbf{n}\alpha} \varphi^0_{\mathbf{m}\beta}. \tag{13}$$

Here, μ is the number of the exciton band, $\mathbf{k}$ is the wave vector of the exciton, and φ^0 and φ^f are the wave functions of the molecules in the ground state and in the f-th excited state (for more detail, see reference 15).

The matrix element of operator (11), as constructed from the functions of the initial and final states, is equal to

$$\langle \mathbf{k}_0 j;\ 0\,|\, H^{(2)} \,|\, \mathbf{k}'_0 j';\ \mu\mathbf{k}\rangle$$
$$= \frac{2e^2\pi\hbar\,(\mathbf{l}_{\mathbf{k}_0 j}\ \mathbf{l}_{\mathbf{k}'_0 j'})}{mcV\sqrt{\ }\ \sqrt{k_0 k'_0}} \sum_{\mathbf{n}\alpha,\,\mathbf{m}\beta} B^*_{\beta\mu}(\mathbf{k})\, F_{\mathbf{m}\beta}(\mathbf{n}, \alpha) \exp\{i(\mathbf{k}_0 - \mathbf{k}'_0,\ \mathbf{r}_{\mathbf{n}\alpha})$$
$$- i\mathbf{k}\mathbf{r}_{\mathbf{m}\beta}\}, \tag{14}$$

where

$$_{\mathbf{n}\beta}(\mathbf{n}, \alpha) = \langle \Phi_0 \,|\, F(\mathbf{n},\alpha) \,|\, \Phi^f_{\mathbf{m}\beta}\rangle. \tag{15}$$

However, since [see Eq. (13)],

$$F_{\mathbf{m}\beta}(\mathbf{n},\alpha) = \langle \varphi^0_{\mathbf{n}\alpha} \,|\, F(\mathbf{n},\alpha) \,|\, \varphi^f_{\mathbf{n}\alpha}\rangle\, \delta_{\mathbf{n}\alpha,\,\mathbf{m}\beta} \equiv F_\alpha\, \delta_{\mathbf{n}\alpha,\,\mathbf{m}\beta}, \tag{16}$$

(here $\delta_{x,y} = 0$ if $x \neq y$, and $\delta_{x,y} = 1$ if $x = y$), we find that

$$\langle \mathbf{k}_0 j;\ 0\,|\, H^{(2)} \,|\, \mathbf{k}'_0 j';\ \mu,\ \mathbf{k}\rangle$$
$$= \frac{2e^2\pi\hbar\,(\mathbf{l}_{\mathbf{k}_0 j}\ \mathbf{l}_{\mathbf{k}'_0 j'})}{mcV\sqrt{k_0 k'_0}} \sqrt{N} \sum_\alpha B^*_{\alpha\mu}(\mathbf{k})\, F_\alpha\, \delta_{\mathbf{k}_0 - \mathbf{k}'_0,\ \mathbf{k}+\mathbf{b}}, \tag{17}$$

where $\mathbf{b}$ is an integer reciprocal-lattice vector.

The Raman-scattering cross section has the form (see reference 14):

$$d\sigma_k = (2\pi V/\hbar c)\,|\langle \mathbf{k}_0 j;\ 0\,|\, H^{(2)} \,|\, \mathbf{k}'_0 j';\ \mu,\ \mathbf{k}_0 - \mathbf{k}'_0\rangle|^2 \rho_{\varepsilon'}, \tag{18}$$

where the number of final states per unit energy interval is

$$\rho_{\varepsilon'} = (k'_0)^2 (2\pi)^{-3} V (dk'_0/d\varepsilon')\, d\Omega = (k'_0)^2 (2\pi)^{-3} (V/\hbar c)\, d\Omega;$$

Here we have taken into account the fact that $\epsilon' = \hbar k'_0 c = \hbar\omega'_0$. Hence, upon substituting Eq. (17) into (18), summing over the polarization j', and averaging over the polarization j, we find that

$$d\sigma_k = r_0^2 \left(\frac{k'_0}{k_0}\right) \frac{1 + \cos^2\theta}{2} N \left|\sum_\alpha B^*_{\alpha\mu}(\mathbf{k})\, F_\alpha\right|^2 d\Omega. \tag{19}$$

If the wavelength of the exciton is large in comparison with the molecular dimensions, then for $\mathbf{b} = 0$:*

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*Exciton states corresponding to sufficiently high quasi-momentum values (see Sec. 1) exhibit basically the expected properties of excitons derived without taking retardation into account.[1-4]

*Processes with $\mathbf{b} \neq 0$ and $k \ll b$ will give rise to inelastically scattered photons concentrated in the neighborhoods of the Bragg maxima (of course, it is assumed that the crystal is properly oriented). However, since in such cases the change in the wave vector of the photons is great, the quantities $|F_\alpha|$ [see Eqs. (12) and (16)] will be considerably lower than in the case $\mathbf{b} = 0$. In addition, since the quantities $|F_\alpha|$ decrease with increasing $|\mathbf{b}|$, the largest cross section will be found in the neighborhood of the first Bragg maximum (the smallest value of $|\mathbf{b}|$). From the standpoint of observing scattering near the Bragg maxima, "loosely-packed" crystals, in which the lattice parameter is considerably greater than the dimensions of the molecules, are of interest.

$$F_\alpha \approx i\,(\mathbf{P}_\mu(\mathbf{k})\,\mathbf{k})/e; \tag{20}$$

Here

$$\mathbf{P}_\mu(\mathbf{k}) = \sum_\alpha B^*_{\alpha\mu}(\mathbf{k})\,\mathbf{P}^{0f}_\alpha$$

is the amplitude of the dipole moment corresponding to the exciton state $\mu\mathbf{k}$, and

$$\mathbf{P}^{0f}_\alpha = \int \varphi^{*0}_{\mathbf{n}\alpha}\,\mathbf{P}_{\mathbf{n}\alpha}\,\varphi^{f}_{\mathbf{n}\alpha}\,d\tau,$$

where $\mathbf{P}_{\mathbf{n}\alpha}$ is the dipole-moment operator of the molecule $\mathbf{n}\alpha$.

Thus,

$$d\sigma_k = (r_0^2/2e^2\Delta)\,(k_0'/k_0)\,(1+\cos^2\theta)\,V\,|\mathbf{P}_\mu(\mathbf{k})|^2\,k^2\cos^2\Psi\,d\Omega, \tag{21}$$

where Δ is the unit-cell volume, and Ψ is the angle between the vectors $\mathbf{P}_\mu(\mathbf{k})$ and $\mathbf{k}$.

Upon applying Eq. (21) as before, we find that

$$\frac{d\sigma_k}{d\sigma_0} = 16\pi\left(\frac{k_0'}{k_0}\right)\left(\frac{\omega_0^2}{\omega_p^2}\right)\left(\frac{|\mathbf{P}_\mu(\mathbf{k})|^2}{mc^2\Delta}\right)\sin^2\frac{\theta}{2}\cos^2\Psi,$$

$$d\sigma_0 = \tfrac{1}{2}\,r_0^2(1+\cos^2\theta)\,\overline{N}_0 V\,d\Omega, \qquad \omega_p^2 = 4\pi e^2\,\overline{N}_0/m. \tag{22}$$

It follows from Eqs. (21) and (22) that the calculated cross-section is proportional to the oscillator strength of the transition in the molecule. Among molecular crystals, the most intense dipole transitions are exhibited by anthracene (the second electronic transition[16]), for which $|\mathbf{P}_\mu|/e \approx 3 \times 10^{-8}$ cm. In this case, assuming that $\Delta \approx 125 \times 10^{-24}$ cm^3, $\theta \approx 3°$, $\cos^2\Psi \approx 1/3$, $k_0'/k_0 \approx 1$, and $\omega_0/\omega_p \approx 5 \times 10^2$, we find that $d\sigma_k/d\sigma_0 \approx 4 \times 10^{-3}$. At $\theta \approx 30°$, the ratio $d\sigma_k/d\sigma_0 \approx 4 \times 10^{-1}$.

It follows from a comparison of Eqs. (22) and (7) that for Frenkel' excitons the quantity a_j [see also Eq. (4)] is inversely proportional to the square of the matrix element of the dipole-moment operator. Also, a_j becomes infinite for those excitons which cannot be excited by light in the dipole approximation. In the latter case, according to Eq. (6), $d\sigma_k \to 0$; in the present approximation this is due to the vanishing of the amplitude of the electric field [see Eq. (4) for $a_j \to \infty$].

Of course, Eq. (22) can be derived by first finding the dielectric-constant tensor $\epsilon_{ij}(\omega, \mathbf{k})$ for our model and then calculating the coefficient a_j entering into Eqs. (4) and (6).

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PHENOMENOLOGICAL DERIVATION OF THE EQUATIONS OF VORTEX MOTION IN He II

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A phenomenological derivation, based upon conservation laws, of the equations of vortex motion in He II is presented, with dissipative processes taken into account.

IN analyzing the oscillations of a stack of discs in rotating He II, Hall[1] introduced elastic forces for the vortex filaments in order to explain the experimental results. L. D. Landau advanced the idea that these forces may be derived phenomenologically with the aid, as usual, of conservation laws. Such a derivation is all the more to be desired in that it also permits the derivation, in a wholly rigorous manner, of the dissipative terms, which, Hall[1] felt could not be expressed in closed form through the velocity of motion of the liquid.*

A paper has recently been published by Mamaladze and Matinyan,[3] in which the problem of the oscillations of a single disc in rotating He II has been solved. The equations used therein, however, are incomplete, in that for $\omega = \text{curl}\, \mathbf{v}_s \neq 0$ the mutual friction forces no longer depend upon the difference $\mathbf{v}_n - \mathbf{v}_s$ alone, as assumed in reference 3.

In the present paper a complete system of hydrodynamic equations is derived for He II in which vortical motion of the superfluid component is taking place, with dissipative processes taken into account.

Let us consider He II in the case of a non-vanishing curl of the superfluid velocity. The fundamental distinction between the motion under consideration and the curl-free case lies in the dependence of the internal energy ϵ of the liquid upon the absolute value of the curl of the velocity. This dependence may be expressed in the differential form:

$$\delta\varepsilon = \lambda\delta\omega, \qquad \omega = |\operatorname{rot} \mathbf{v}_s|. \tag{1}^\dagger$$

The coefficient λ, as follows from a microscopic treatment of the vortex filaments, is given by

$$\lambda = \rho_s \frac{h}{m} \ln \frac{R}{a},$$

where ρ_s is the superfluid component density, m is the mass of the He^3 atom, and R/a is the ratio of the distance between vortices to the effective radius of a vortex.

We shall henceforth follow the standard method, proceeding from the conservation laws.

In the momentum flux tensor Π_{ik} and in the energy flux $\mathbf{Q}$ there appear additional terms which we shall designate π_{ik} and $\mathbf{q}$. The mass current vector $\mathbf{j}$, which equals the momentum per unit volume of the liquid, can, as usual, be defined in such a way that the continuity equation retains its form. Taking this into account, the equations for conservation of mass, energy, and momentum will be

$$\partial\rho/\partial t + \operatorname{div} \mathbf{j} = 0, \tag{2}$$

$$\partial E/\partial t + \operatorname{div} (\mathbf{Q}_0 + \mathbf{q}) = 0, \tag{3}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_k}(\Pi^0_{ik} + \pi_{ik}) = 0, \tag{4}$$

where ρ is the density of the liquid, and E is the energy per unit volume of the He II. The latter can, with the aid of a Galilean transformation, be expressed as an internal energy ϵ of the liquid in a reference system moving with the superfluid component velocity $\mathbf{v}_s$:

$$E = \tfrac{1}{2}\rho v_s^2 + \mathbf{p}\mathbf{v}_s + \varepsilon, \tag{5}^*$$

where $\mathbf{p}$ is the momentum of the liquid per unit volume in this same system. The total momentum per unit volume, in the stationary coordinate system, is also expressed in terms of $\mathbf{p}$:

$$\mathbf{j} = \mathbf{p} + \rho\mathbf{v}_s. \tag{6}$$

The internal energy ϵ obeys a thermodynamic identity which must take into account the previously-indicated (1) change in the energy of the liquid due to the vortex motion:

$$d\varepsilon = TdS + \mu d\rho + (\mathbf{v}_n - \mathbf{v}_s\,,\, d\mathbf{p}) + \lambda\, d\omega, \tag{7}$$

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*While the present paper was being prepared for publication, there appeared a review by Hall,[2] in which he has derived, by less general means, an expression analogous to ours for the mutual friction force. His final result, however, differs from ours, evidently as a result of a mathematical error of his.

†rot = curl.

*$\mathbf{pv}$ or $(\mathbf{p}, \mathbf{v}) = \mathbf{p}\cdot\mathbf{v}$.

where S and T are the entropy and temperature, μ is the chemical potential, and $\mathbf{v}_n$ is the normal component velocity.

We shall take the undisturbed momentum flux tensor and energy flux in the form[4]:

$$\mathbf{Q}_0 = (\mu + \tfrac{1}{2} v_s^2)\,\mathbf{j} + ST\mathbf{v}_n + \mathbf{v}_n(\mathbf{v}_n\mathbf{p}), \tag{8}$$

$$\Pi^0_{ik} = \rho v_{si} v_{sk} + v_{si} p_k + v_{nk} p_i + p\delta_{ik}, \tag{9}$$

where ρ is the pressure:

$$p = -\varepsilon + TS + \mu\rho + (\mathbf{v}_n - \mathbf{v}_s,\ \mathbf{p}). \tag{10}$$

We thus include all of the dissipative processes in $\mathbf{q}$ and π_{ik}.

In addition to the conservation equations (2) – (4), the system of hydronamic equations includes the equation of motion of the superfluid component and the entropy growth equation. Taking into account the additional terms arising from the vortex motion, we have

$$\partial\mathbf{v}_s/\partial t + (\mathbf{v}_s\nabla)\,\mathbf{v}_s + \nabla\mu = \mathbf{f}, \tag{11}$$

$$\partial S/\partial t + \operatorname{div} S\mathbf{v}_n = R/T, \tag{12}$$

where the quantities $\mathbf{f}$ and R are to be determined (R is the dissipative function).

In order to determine the form of the unknown functions let us differentiate the left- and right-hand parts of (5) with respect to time. Substituting the time derivatives from (2), (4), and (11), we obtain

$$\begin{aligned} E = &-\operatorname{div} Q_0 - \operatorname{div}(\pi\mathbf{v}_n) + T(\dot{S} + \operatorname{div} S\mathbf{v}_n) \\ &+ \lambda\dot{\omega} + \lambda v_{ni}\,\partial\omega/\partial x_i + \pi_{ik}\,\partial v_{ni}/\partial x_k \\ &+ (\mathbf{j} - \rho\mathbf{v}_n,\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]), \end{aligned} \tag{13}$$

where

$$(\pi\mathbf{v}_n)_i = \pi_{ik} v_{nk}.$$

Introducing the notation $\boldsymbol{\nu} = \boldsymbol{\omega}/\omega$ and substituting into (13) the expression

$$\lambda\dot{\omega} = \lambda\boldsymbol{\nu}\operatorname{rot}\dot{\mathbf{v}}_s = \lambda\boldsymbol{\nu}\operatorname{rot}\{\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]\} - \lambda\boldsymbol{\nu}\operatorname{rot}[\boldsymbol{\omega}\mathbf{v}_n], \tag{14}^*$$

we obtain, after certain transformations which separate out the terms having the form of divergences:

$$\begin{aligned} \dot{E} &+ \operatorname{div}\{\mathbf{Q}_0 + (\pi\mathbf{v}_n) + \lambda[\boldsymbol{\nu},\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]]\} \\ &= T(\dot{S} + \operatorname{div} S\mathbf{v}_n) + (\pi_{ik} - \lambda\omega\delta_{ik} + \lambda\omega_i\omega_k/\omega)\,\partial v_{ni}/\partial x_k \\ &+ (\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s],\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}). \end{aligned} \tag{15}$$

Comparing this expression with the equations for conservation of energy (3) and entropy growth (12), we find

$$\mathbf{q} = (\pi\mathbf{v}_n) + \lambda[\boldsymbol{\nu},\ \mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s]], \tag{16}$$

$$\begin{aligned} R = &-(\pi_{ik} - \lambda\omega\delta_{ik} + \lambda\omega_i\omega_k/\omega)\,\partial v_{ni}/\partial x_k \\ &- (\mathbf{f} + [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s],\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}). \end{aligned} \tag{17}$$

Requiring that the dissipative function be a positive definite quadratic form, we obtain from (17) the most general expressions for the vector terms

$$\begin{aligned} \mathbf{f} = &-[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s] + \alpha[\boldsymbol{\omega},\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}] \\ &+ \beta[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}]] - \gamma\boldsymbol{\nu}(\boldsymbol{\omega}, \mathbf{j} - \rho\mathbf{v}_n + \operatorname{rot}\lambda\boldsymbol{\nu}) \end{aligned} \tag{18}$$

($\beta, \gamma \geq 0$, as a consequence of the condition $R \geq 0$), and for the tensor terms

$$\pi_{ik} = \lambda\omega\delta_{ik} - \lambda\omega_i\,\omega_k/\omega + \tau_{ik}. \tag{19}$$

τ_{ik} appears as a viscous stress tensor and must be expressed in general form with the aid of the viscosity tensor η_{iklm}:

$$\tau_{ik} = \eta_{iklm}\,\partial v_{nl}/\partial x_m. \tag{20}$$

When vortical motion takes place in the liquid, with ω assuming a specific orientation, the viscosity tensor may in principle contain additional terms, as compared with the vortex-free case. In precisely this way the momentum of relative motion $\mathbf{p}$ - which in vortex-free flow is directed, due to the isotropy of the liquid, along the relative velocity

$$\mathbf{p} = \rho_n(\mathbf{v}_n - \mathbf{v}_s), \tag{21}$$

where ρ_n is the normal density - may now have as well a component along the second, distinct direction ω.

In principle, transfer of entropy and momentum by the vortices is also possible.

All of these phenomena, however, are quadratic effects and are extremely small. In particular, as in the absence of vortical motion, the tensor η_{iklm} reduces to the coefficients of first and second viscosity (which are independent of ω). The thermal conductivity coefficient also remains unchanged, as do the dissipative gradient terms in the superfluidity equation (in our derivation, $\mathbf{f}$ is determined accurate to the gradient terms).

Substituting into the right-hand part of (18) the expression

$$\mathbf{j} - \rho\mathbf{v}_n = \mathbf{p} + \rho(\mathbf{v}_s - \mathbf{v}_n) = -\rho_s(\mathbf{v}_n - \mathbf{v}_s), \tag{22}$$

which is obtained with the aid of Eq. (21), we may transform $\mathbf{f}$ into the form

$$\begin{aligned} \mathbf{f} = &-\rho_s^{-1}[\boldsymbol{\omega}\operatorname{rot}\lambda\boldsymbol{\nu}] - (1 + \alpha\rho_s)[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}] \\ &- \beta\rho_s[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}]] \\ &+ \gamma\rho_s\boldsymbol{\nu}(\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\operatorname{rot}\lambda\boldsymbol{\nu}). \end{aligned} \tag{23}$$

Let us now write out the superfluidity equation

$^*[\boldsymbol{\omega}\mathbf{v}_n] = \boldsymbol{\omega} \times \mathbf{v}_n;\ [\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s] = \boldsymbol{\omega} \times (\mathbf{v}_n - \mathbf{v}_s)$.

$$\partial \mathbf{v}_s/\partial t + (\mathbf{v}_s \nabla)\,\mathbf{v}_s + \nabla\mu = -\rho_s^{-1}[\boldsymbol{\omega}\,\mathrm{rot}\,\lambda\boldsymbol{\nu}]$$
$$-\beta'\rho_s[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\,\mathrm{rot}\,\lambda\boldsymbol{\nu}]$$
$$-\beta\rho_s[\boldsymbol{\nu}[\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\,\mathrm{rot}\,\lambda\boldsymbol{\nu}]]$$
$$+\gamma\rho_s\boldsymbol{\nu}(\boldsymbol{\omega},\ \mathbf{v}_n - \mathbf{v}_s - \rho_s^{-1}\,\mathrm{rot}\,\lambda\boldsymbol{\nu}), \qquad (24)$$

where $\beta'\rho_S$ represents $1 + \alpha\rho_S$. The first term on the right coincides with that derived by Hall in reference 1. The remaining three terms describe the mutual friction force. For the case of straight-line vortices $\nu \equiv \omega/\omega = \text{const.}$, from which it is easy to derive the relation between the coefficients β and β' and the coefficients of Hall and Vinen:

$$\beta = \tfrac{1}{2}B\rho_n/\rho\rho_s, \quad \beta' = \tfrac{1}{2}B'\rho_n/\rho\rho_s.$$

In the case of transverse deflections of the vortices, the mutual friction includes additional terms in curl $\lambda\nu$. Under certain conditions these may play a predominant role.

The term having the coefficient γ represents a longitudinal (along ω) mutual friction force which, it appears, may arise when the direction of the individual vortex filaments deviates from the direction of the mean curl of the velocity ω; for example, in the presence of thermal oscillations. In view of the smallness of these effects, however, the coefficient γ is evidently extremely small by comparison with β and β'. The increments to the momentum flux tensor resulting from the vortex motion have the form

$$\tilde{\pi}_{ik} = \lambda\omega\delta_{ik} - \lambda\omega_i\omega_k/\omega \qquad (25)$$

and lead to renormalization of the pressure to the value $\lambda\omega$, and to the appearance of a term $-\lambda\omega_i\omega_k/\omega$, which may be interpreted as the vortex filament tension.

Let us now consider the change in the energy flux. Substituting into Eq. (16) $\tilde{\pi}_{ik}$ from (25) we obtain

$$\mathbf{q} = \lambda[\boldsymbol{\nu}, \mathbf{f} + [\mathbf{v}_s\boldsymbol{\omega}]]. \qquad (26)$$

This expression is especially easy to interpret in the absence of any longitudinal component of $\mathbf{f}(\gamma = 0)$. In this case we obtain from (24), taking the curl of both sides

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \mathrm{rot}\{\mathbf{f} + [\mathbf{v}_s\boldsymbol{\omega}]\} = \mathrm{rot}\,[\mathbf{v}_L\boldsymbol{\omega}], \qquad (27)$$

which has the form of a transport equation for the vector ω, with a velocity

$$\mathbf{v}_L = \mathbf{v}_s - \frac{1}{\rho_s}\,\mathrm{rot}\,\lambda\boldsymbol{\nu} + \beta'\rho_s(\mathbf{v}_n - \mathbf{v}_s - \mathrm{rot}\,\lambda\boldsymbol{\nu})$$
$$-\beta\rho_s\left[\boldsymbol{\nu};\ \mathbf{v}_n - \mathbf{v}_s - \frac{1}{\rho_s}\mathrm{rot}\,\lambda\boldsymbol{\nu}\right],$$

This may be called the velocity of movement of the vortex filaments (and is accurate to terms parallel to the vector ω, which is adequate in the absence of longitudinal friction).

Thus, as follows from (27), the energy flux may be written in the form

$$\mathbf{q} = \omega\,\frac{\partial\varepsilon}{\partial\omega}[\boldsymbol{\nu}[\mathbf{v}_L, \boldsymbol{\nu}]],$$

which represents energy transport by the vortices in a direction perpendicular to ω.

Finally, we shall consider the boundary conditions which must be obeyed by the superfluid component at the surface of a solid and at a free surface as the normal component velocity falls to zero. We shall limit our treatment to the vortices terminating at the surface.

It is clear that in the case of a free surface the tangential components of the tension in the liquid must be zero; i.e., $(\pi\mathbf{N})_t = 0$, whence $\omega_N \cdot \omega_t = 0$. The vortices are thus perpendicular to the free surface of the liquid ($\mathbf{N}$ is the normal vector).

In the case of a very rough surface the velocity of the vortices at the surface must coincide with that of the surface itself.

In the event of vortex slippage, however, intensive energy dissipation must take place, as observed by Hall.[1]

The mechanical energy of the vortex motion dissipated at the surface is numerically equal to the difference between the energy flux directed toward the surface and the mechanical energy actually transferred to the surface, the latter being equal to the work done by the tension forces as the wall moves at a velocity $\mathbf{u} = \mathbf{v}_n|_{\text{surface}}$;

$$-\dot{E}_{\mathrm{mech}} = \mathbf{qN} - (\tilde{\pi}\mathbf{u})\,\mathbf{N}. \qquad (28)$$

Here, the vector $(\pi\mathbf{u})$ is determined in accordance with (13), and $\mathbf{N}$ is the normal vector external to the liquid.

Substituting q and $\tilde{\pi}_{ik}$ from (25) and (26), after interchanging the order of the factors in the mixed vector product

$$-\dot{E}_{\mathrm{mech}} = \lambda(\mathbf{f} + [\mathbf{v}_s - \mathbf{u}, \boldsymbol{\omega}],\ [\mathbf{N}\boldsymbol{\nu}]), \qquad (29)$$

from which it follows that

$$\mathbf{f} + [\mathbf{v}_s - \mathbf{u},\ \boldsymbol{\omega}]\,|_{\mathrm{sur}} = \zeta[\mathbf{N}\boldsymbol{\omega}] + \zeta'[[\mathbf{N}\boldsymbol{\nu}]\boldsymbol{\omega}]. \qquad (30)$$

From the qualitative microscopic treatment we obtain the values for the coefficients ζ and ζ':

$$\zeta/B \approx \zeta'/B' \approx \rho_n h/\rho\ md, \qquad (31)$$

where d is the mean size of the surface irregularities.

The boundary condition (30) may be rewritten in terms of the vortex velocity (for $\gamma = 0$):

$$\mathbf{v}_L - \mathbf{u} = \zeta[\boldsymbol{\nu}[\mathbf{N}\boldsymbol{\nu}]] + \zeta'[\mathbf{N}\boldsymbol{\nu}], \qquad (32)$$

which for $\zeta' = 0$ agrees with the boundary condition employed by Hall.[1] The transitions $\zeta, \zeta' \to 0$ and $\zeta, \zeta' \to \infty$ correspond to transitions to an absolutely rough surface and to a free surface, respectively.

We shall now write down the complete system of equations for an incompressible fluid. Let us determine the μ appearing in the superfluidity equation. Making use of the thermodynamic identity (7) and the definition of p (10) we find, if we take into account the fact that λ is weakly dependent upon ω,

$$\mu = \mu(p, T) - (\rho_n/2p)\,(\mathbf{v}_n - \mathbf{v}_s)^2 + \lambda\omega/\rho.$$

Moreover, introducing the renormalized pressure $p_0 = p + \lambda\omega$, as follows from (25), we have, finally

$$\mu = \mu(p_0 - \lambda\omega, T) + \frac{\lambda\omega}{\rho} - \frac{\rho_n}{2\rho}(\mathbf{v}_n - \mathbf{v}_s)^2$$

$$\approx \mu(p_0, T) - \frac{\rho_n}{2\rho}(\mathbf{v}_n - \mathbf{v}_s)^2. \tag{33}$$

The full system of equations of motion for an incompressible liquid ($\operatorname{div}\mathbf{v}_n = \operatorname{div}\mathbf{v}_s = 0$) consists of the superfluidity equation (24), in which $\nabla\mu$ must be replaced by $p_s = \rho_s p/\rho$, and the equation of motion of the normal component (obtained from the momentum conservation equation):

$$\begin{aligned}\rho_n[\partial\mathbf{v}_n/\partial t + (\mathbf{v}_n\nabla)\,\mathbf{v}_n] &= -\nabla p_n + \eta\Delta\mathbf{v}_n\\ &- \tfrac{1}{2}B\rho_s\rho_n\rho^{-1}[\boldsymbol{\omega}[\boldsymbol{\nu}, \mathbf{v}_n - \mathbf{v}_s - \lambda\rho_s^{-1}\operatorname{rot}\boldsymbol{\nu}]]\\ &- \tfrac{1}{2}B'\rho_s\rho_n\omega^{-1}[\boldsymbol{\omega}, \mathbf{v}_n - \mathbf{v}_s - \lambda\rho_s^{-1}\operatorname{rot}\boldsymbol{\nu}],\\ p_n &= p\rho_n/\rho,\end{aligned} \tag{34}$$

under the boundary conditions

$$\begin{gathered}\mathbf{v}_n|_{\text{sur}} = \mathbf{u}_{\text{sur}}, \qquad (\mathbf{v}_s - \mathbf{u}, \mathbf{N})_{\text{sur}} = 0,\\ \mathbf{f} + [\mathbf{u} - \mathbf{v}_s, \boldsymbol{\omega}] = \zeta[\mathbf{N}\boldsymbol{\omega}] + \zeta'[[\mathbf{N}\boldsymbol{\nu}]\,\boldsymbol{\omega}].\end{gathered} \tag{35}$$

It is possible, using the hydrodynamic equations thus obtained, to compute the moment of the forces acting upon a disc oscillating in rotating He II.

If the frequency of the disc oscillations is represented by Ω, and the angular frequency of rotation of the liquid by ω_0, then, with accuracy out to $B\rho_n\omega_0/\rho\Omega$, the motion of the normal component will not influence the motion of the superfluid. It also turns out that, to the same degree of accuracy, the change in the moment of the forces acting on the disc on the part of the normal component $\eta B/\lambda \sim 1/30$ of the moment of the forces due to vortex tension. We shall now present the expression for the moment of the forces acting on a disc at a distance l below the free surface of the liquid (these calculations were carried out to the accuracy indicated above):

$$\Delta M = M_s = \alpha_0\pi R^4\lambda\omega_0\left(\frac{N_1}{1 - (\zeta' + i\zeta)\,N_1/\Omega} + \frac{N_2}{1 + (\zeta' - i\zeta)\,N_2/\Omega}\right), \tag{36}^*$$

where

$$N_i = k_i \operatorname{tg} k_i l \quad (i = 1, 2),$$

$$k_1 = \sqrt{\frac{\Omega - 2\omega_0}{\lambda}}\left(1 + i\frac{B\rho_n}{4\rho}\right),$$

$$k_2 = i\sqrt{\frac{\Omega + 2\omega_0}{\lambda}}\left(1 + i\frac{B\rho_n}{4\rho}\right),$$

α_0 is the amplitude of disc oscillations, and R is the radius of the disc. In this approximation, Eq. (36) agrees, for the condition $l \to \infty$ with that obtained by Mamaladze and Matinyan[3] (it would be improper to take into consideration the higher degree of precision indicated in reference 3, in view of the incompleteness of the equations used therein).

In conclusion, the authors express their deep gratitude to Academician L. D. Landau for his consideration of the problem under review and for his valued advice.

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*tg = tan.

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ON GAUGE TRANSFORMATIONS OF GREEN'S FUNCTIONS

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The law is obtained for the transformation of many-particle Green's functions under changes of gauge. General identities of the type of Ward's identity follow from this law.

1. INTRODUCTION

THE paper by Landau and Khalatnikov[1] on the laws of gauge transformation of the Green's functions $\langle T\psi(x)\bar{\varphi}(y)\rangle$ and $\langle T\psi(x)\bar{\varphi}(y)A_\mu(u)\rangle$ has been followed by a number of papers connected with this question.[2-5]

The present paper proves a general theorem which establishes the law of gauge transformation for Green's functions with an arbitrary number of operators of arbitrary charged and neutral fields, without choosing a specific form of the interaction.

The laws of gauge transformations are determined only by the transformation properties of the Heisenberg operators under changes of gauge, independent of special forms of the interactions.

In electrodynamics the law of transformation of somewhat differently defined many-particle Green's functions has been derived recently by Okubo.[4] His proof, however, is valid only in electrodynamics and is connected with the expansion of the Green's functions in perturbation-theory series.

By means of the law for gauge transformations of Green's functions one can give a derivation of identities of the type of the Ward identity which is independent of gauge and is the most general and natural proof of these relations. This has also been done for the connection between the vertex part and the one-particle Green's function (cf. references 4 and 5). This method is used here to obtain general identities connecting many-particle Green's functions, and some concrete examples of identities of the Ward type are presented.

2. DEFINITIONS OF THE GREEN'S FUNCTIONS

We present the definitions of the many-particle Green's functions for which we shall consider the laws of gauge transformations. The Green's function without photon ends is defined as the vacuum expectation value of the T product,

$$G(x^1\ldots x^n,\ y^1\ldots y^n,\ z^1\ldots) = \langle T\Psi\rangle \equiv \langle T\psi_1(x^1)\ldots\psi_n(x^n)\varphi_1(y^1)\ldots\varphi_n(y^n)\chi_1(z^1)\ldots\rangle, \quad (1)$$

where $\psi(x)$ and $\varphi(y)$ are operators for arbitrary lepton, meson, and baryon charged fields, which transform under gauge transformation by the law

$$\psi'(x) = \exp\,[ie\Lambda(x)]\psi(x), \qquad \varphi'(y) = \varphi(y)\exp\,[-ie\Lambda(y)], \quad (2)$$

and $\chi(z)$ are operators of arbitrary neutral fields other than the electromagnetic field.

We introduce the Green's functions that include electromagnetic-field operators $A_\mu(u)$ in the following way:

$$G_\mu(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u) = \langle T\Psi A_\mu(u)\rangle, \quad (3)$$

$$G_{\mu_1\mu_2}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1u^2) = \langle T\Psi A_{\mu_1}(u^1)A_{\mu_2}(u^2)\rangle - \langle T\Psi\rangle\langle TA_{\mu_1}(u^1)A_{\mu_2}(u^2)\rangle \quad (4)$$

. . . .

$$\begin{aligned} G_{\mu_1\mu_2\ldots\mu_m}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1u^2\ldots u^m) &= \langle T\Psi A_{\mu_1}(u^1)A_{\mu_2}(u^2)\ldots A_{\mu_m}(u^m)\rangle \\ &- \sum_{k>l} G_{\mu_1\ldots\mu_{l-1}\mu_{l+1}\ldots\mu_{k-1}\mu_{k+1}\ldots\mu_m}\langle TA_{\mu_l}(u^l)A_{\mu_k}(u^k)\rangle \\ &- \sum_{i>j>k>l} G_{\mu_1\ldots\mu_{l-1}\mu_{l+1}\ldots\mu_{k-1}\mu_{k+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_m} \\ &\times \langle TA_{\mu_i}(u^i)A_{\mu_j}(u^j)A_{\mu_k}(u^k)A_{\mu_l}(u^l)\rangle - \ldots. \end{aligned} \quad (5)$$

These expressions can be obtained formally by means of functional differentiation with respect to an external current:

$$G_{\mu_1\mu_2\ldots\mu_m}(x^1\ldots,\ y^1\ldots,\ z^1\ldots,\ u^1\ldots u^m) = \left.\frac{\delta^m\langle T\Psi\rangle}{\delta J_{\mu_1}(u^1)\,\delta J_{\mu_2}(u^2)\ldots\delta J_{\mu_m}(u^m)}\right|_{J=0}, \quad (6)$$

where the functional derivative is defined by the rule (Schwinger[6])

$$\frac{\delta\langle T\Psi\rangle}{\delta J_\mu(u)} = \langle T\Psi A_\mu(u)\rangle - \langle T\Psi\rangle\langle A_\mu(u)\rangle, \quad (7)$$

and for $J = 0$ the vacuum expectation value of an odd number of factors $A_\mu(u)$ goes to zero.

The Green's functions defined by the formulas (3) — (5) transform more simply under changes of gauge than do vacuum expectation values, and for the present functions one gets the natural generalization of Ward's identity. In perturbation theory this definition corresponds to the omission of diagrams that contain unconnected parts in which there are only photon external ends.

3. THE TRANSFORMATION OF THE GREEN'S FUNCTIONS UNDER CHANGES OF GAUGE

Let us express the Green's function (5) in an arbitrary gauge in terms of the Green's function in the true gauge of Landau and Khalatnikov (L.K.), in which by definition the photon Green's function has no longitudinal part and the commutator of electromagnetic-field operators at equal times is zero.[5]

The change from the Heisenberg operators ψ^T, φ^T, χ^T, and A_μ^T in the L.K. gauge to the operators ψ, φ, χ, and A_μ in an arbitrary gauge is made according to the rules

$$\psi(x) = \exp[ie\Lambda(x)]\,\psi^\tau(x),\quad \varphi(y) = \varphi^\tau(y)\exp[-ie\Lambda(y)],$$
$$\chi(z) = \chi^\tau(z),\qquad A_\mu(u) = A_\mu^\tau(u) + \partial\Lambda(u)/\partial u_\mu, \tag{8}$$

where $\Lambda(x)$ is a suitably chosen Hermitian operator, which can be represented in the form[1]

$$\Lambda(x) = \int d^4k\lambda(k^2)\,(a_k e^{ikx} + \overset{+}{a}_k e^{-ikx}) \tag{9}$$

(a_k and a_k^+ are creation and annihilation operators).

We may assume that the operator Λ acts in a different Hilbert space from that in which the operators ψ^T, φ^T, χ^T, and A_μ^T act,[1,5] and thus commutes with them. We note also that when we change to a gauge in which the equal-time commutators of the electromagnetic fields are zero by definition (we shall call these true gauges) the choice of Λ is restricted by the condition[5]

$$[\Lambda(\mathbf{x},0),\ \dot\Lambda(0)] = 0. \tag{10}$$

Then the photon Green's function in an arbitrary true gauge can be expressed in the following way:

$$D^c_{\mu\nu}(u^1-u^2) = \langle TA_\mu(u^1)A_\nu(u^2)\rangle = \langle TA_\mu^\tau(u^1)A_\nu^\tau(u^2)\rangle$$
$$+\frac{\partial^2}{\partial u^1_\mu\partial u^2_\nu}\langle T\Lambda(u^1)\Lambda(u^2)\rangle = \int d^4q\exp\{iq(u^1-u^2)\}$$
$$\times[q^2\delta_{\mu\nu} - q_\mu q_\nu)\,d_t(q^2) + q_\mu q_\nu d_l(q^2)]. \tag{11}$$

Let us now find the transformation law for the Green's function (1). According to what has been said,

$$G(x^1\ldots, y^1\ldots, z^1\ldots)$$
$$= G^\tau(x^1\ldots, y^1\ldots, z^1)\,\langle T\exp(ie\Phi)\rangle, \tag{12}$$

where

$$\Phi = \Lambda(x^1) + \ldots + \Lambda(x^n) - \Lambda(y^1) - \ldots - \Lambda(y^n). \tag{13}$$

Owing to the choice of the operator $\Lambda(x)$ in the form (9) the expression $\langle T\exp(ie\Phi)\rangle$ can be calculated by means of Wick's theorem, and furthermore the theorem can be applied directly to Φ, since Φ depends linearly on Λ. Thus

$$\langle T\exp(ie\Phi)\rangle = \sum_{k=0}^{\infty}\frac{(ie)^{2k}}{(2k)!}\langle T\Phi^{2k}\rangle$$
$$= \sum_{k=0}^{\infty}\frac{(ie)^{2k}}{(2k)!}(2k-1)!!\,\langle T\Phi^2\rangle^k = \exp\left(-\frac{e^2}{2}\langle T\Phi^2\rangle\right), \tag{14}$$

and the transformation law for G is of the form

$$G = \exp(e^2\rho)\,G^\tau, \tag{15}$$

$$\rho = \rho(x^1\ldots, y^1\ldots) = -\tfrac{1}{2}\langle T\Phi^2\rangle$$
$$= \frac{1}{2}\sum_{i,j=1}^{n}[\langle T\Lambda(x^i)\Lambda(y^j)\rangle$$
$$-\langle T\Lambda(x^i)\Lambda(x^j)\rangle - \langle T\Lambda(y^i)\Lambda(y^j)\rangle]. \tag{16}$$

The Green's function $G_{\mu_1\mu_2\ldots\mu_m}$ in an arbitrary gauge is expressed in terms of the Green's function in the L.K. gauge according to the law

$$G_{\mu_1\mu_2\ldots\mu_m} = \exp(e^2\rho)\Big\{G^\tau_{\mu_1\ldots\mu_m} + ie\sum_{r=1}^{m}\rho_{\mu_r}G^\tau_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}$$
$$+ (ie)^2\sum_{r_2>r_1=1}^{m}\rho_{\mu_{r_1}}\rho_{\mu_{r_2}}G^\tau_{\mu_1\ldots\mu_{r_1-1}\mu_{r_1+1}\ldots\mu_{r_2-1}\mu_{r_2+1}\ldots\mu_m} + \ldots$$
$$+ (ie)^k\sum_{r_k>r_{k-1}\ldots>r_1=1}^{m}\rho_{\mu_{r_1}}\rho_{\mu_{r_2}}\ldots\rho_{\mu_{r_k}}$$
$$\times G^\tau_{\mu_1\ldots\mu_{r_1-1}\mu_{r_1+1}\ldots\mu_{r_k-1}\mu_{r_k+1}\ldots\mu_m} + \ldots$$
$$+ (ie)^m\rho_{\mu_1}\rho_{\mu_2}\ldots\rho_{\mu_m}G^\tau\Big\}, \tag{17}$$

where

$$\rho_{\mu_r} \equiv \rho_{\mu_r}(x^1\ldots, y^1\ldots, u^r) = \left\langle T\frac{\partial\Lambda(u^r)}{\partial u^r_{\mu_r}}\Phi\right\rangle$$
$$= \sum_{j=1}^{n}\left\langle T\frac{\partial\Lambda(u^r)}{\partial u^r_{\mu_r}}[\Lambda(x^j) - \Lambda(y^j)]\right\rangle. \tag{18}$$

For the proof we note that we can represent the definition (6) of the many-particle Green's functions in the form

$$G_{\mu_1\mu_2\ldots\mu_m} = \left.\frac{\delta^m\langle T\Psi\rangle}{\delta J_{\mu_1}(u^1)\,\delta J_{\mu_2}(u^2)\ldots\delta J_{\mu_m}(u^m)}\right|_{J=0}$$
$$= \prod_{r=1}^{m}\left(\frac{\delta}{\delta J^\tau_{\mu_r}} + \frac{1}{i\alpha}\frac{\partial}{\partial u^r_{\mu_r}}\right)$$
$$\times\langle T\Psi^\tau\rangle\left.\frac{\left\langle T\exp\left[ie\Phi + i\alpha\sum_{l=1}^{m}\Lambda(u^l)\right]\right\rangle}{\left\langle T\exp\left[i\alpha\sum_{l=1}^{m}\Lambda(u^l)\right]\right\rangle}\right|_{J^\tau=0,\ \alpha=0}, \tag{19}$$

where the functional derivative $\delta/\delta J_\mu^T$ is defined by the rule (7) with A_μ replaced by A_μ^T, and acts only on $\langle T\Psi^T\rangle$. We note that as long as J^T and the parameter α are not set equal to zero the operator $\delta/\delta J_\mu^T + (i\alpha)^{-1}\partial/\partial u_\mu$ gives $A_\mu(u)$ in the form of the sum $A_\mu^T + \partial\Lambda/\partial u_\mu$, according to the rule (7), in which we have only to replace the average $\langle T\ldots\rangle$ by the average

$$\langle T \ldots \exp\,[i\alpha\Sigma\Lambda\,(u^l)]\rangle / \langle T \exp\,[i\alpha\Sigma\Lambda\,(u^l)]\rangle.$$

After calculations analogous to Eq. (14) we get

$$\frac{\langle T \exp\,[ie\Phi + i\alpha\Sigma\Lambda\,(u^l)]\rangle}{\langle T \exp\,[i\alpha\Sigma\Lambda\,(u^l)]\rangle} = \exp\left[e^2\rho - \alpha e\sum_{l=1}^{m}\langle T\Phi\Lambda\,(u^l)\rangle\right]. \tag{20}$$

Differentiation of the last expression obviously gives

$$\left(\frac{1}{i\alpha}\right)^k \frac{\partial^k}{\partial u_{\mu_1}^1 \ldots \partial u_{\mu_k}^k} \exp\left[e^2\rho - \alpha e\sum_{l=1}^{m}\langle T\Phi\Lambda\,(u^l)\rangle\right]\Bigg|_{\alpha=0}$$

$$= (ie)^k \rho_{\mu_1}\rho_{\mu_2}\ldots\rho_{\mu_k} \exp\,(e^2\rho). \tag{21}$$

Since

$$G^\tau_{\mu_1\mu_2\ldots\mu_k} = \frac{\delta^m\,\langle T\Psi^\tau\rangle}{\delta J^\tau_{\mu_1}(u^1)\,\delta J^\tau_{\mu_2}(u^2)\ldots\delta J^\tau_{\mu_k}(u^k)}\Bigg|_{J^\tau=0} \tag{22}$$

$$\prod_{r=1}^{m}\left(\frac{\delta}{\delta J^\tau_{\mu_r}} + \frac{1}{i\alpha}\frac{\partial}{\partial u^r_{\mu_r}}\right) = \frac{\delta^m}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_m}}$$

$$+ \frac{1}{i\alpha}\sum_{r=1}^{m}\frac{\delta^{m-1}}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_{r-1}}\delta J^\tau_{\mu_{r+1}}\ldots\delta J^\tau_{\mu_m}}\frac{\partial}{\partial u^r_{\mu_r}}$$

$$+ \left(\frac{1}{i\alpha}\right)^2 \sum_{r_2>r_1=1}^{m} \frac{\delta^{m-2}}{\delta J^\tau_{\mu_1}\ldots\delta J^\tau_{\mu_{r_1-1}}\delta J^\tau_{\mu_{r_1+1}}\ldots\delta J^\tau_{\mu_{r_2-1}}\delta J^\tau_{\mu_{r_2+1}}\ldots\delta J^\tau_{\mu_m}}$$

$$\times \frac{\partial^2}{\partial u^{r_1}_{\mu_{r_1}}\partial u^{r_2}_{\mu_{r_2}}} + \ldots + \left(\frac{1}{i\alpha}\right)^m \frac{\partial^m}{\partial u^1_{\mu_1}\ldots\partial u^m_{\mu_m}}, \tag{23}$$

the transformation law (17) follows from the expression (19).

We shall make two comments. First, we note that Eq. (17) can also be proved without the functional differentiation (6), (7) by the method of mathematical induction, if we use the definition (5) of the Green's functions. Second, we emphasize that in the proof of the theorem essential use has been made of the definition (9), which enables us to apply Wick's theorem directly to the operators Λ. If we renounce this definition, the law in question is not true. For example, if we set*

$$\Lambda(x) = \Lambda_1(x)\Lambda_2(x), \tag{24}$$

where $\Lambda_1(x)$ and $\Lambda_2(x)$ are Hermitian operators of the form (9) that act in different Hilbert spaces, and take for simplicity

$$\langle T\Lambda_1(x)\Lambda_1(y)\rangle_1 = \langle T\Lambda_2(x)\Lambda_2(y)\rangle_2 = F(x-y), \tag{25}$$

then the transformation law of the one-particle Green's function is

$$\langle T(\psi x)\overline{\psi}(y)\rangle = \langle T\psi^\tau(x)\overline{\psi}^\tau(y)\rangle\{1 + e^2[F^2(0)$$

$$- F^2(x-y)]\}^{-1}. \tag{26}$$

In this connection we note that the derivation of the law (15) for $\langle T\psi(x)\overline{\psi}(y)\rangle$ from group-theory arguments in the paper of Evans, Feldman. and Matthews[5] is erroneous, since the infinitesimal transition operator has been found only from a special (L.K.) gauge, and not from an arbitrary gauge. Therefore the passage from Eq. (3.15) to Eq. (3.16) in reference 5 is illegitimate, in spite of the fact that gauge transformations actually do form a Lie group. Thus in the example just given Eq. (3.15) holds and Eq. (3.16) does not. Equation (3.16) holds only in the case of operators $\Lambda(x)$ of the form (9), but to establish it one actually needs to know the final result.

4. GENERAL IDENTITIES OF THE WARD TYPE

In the L.K. gauge

$$\frac{\partial}{\partial u^r_{\mu_r}} G^\tau_{\mu_1\ldots\mu_r\ldots\mu_k} = 0 \tag{27}$$

in consequence of the Lorentz condition[5]

$$\frac{\partial A^\tau_\mu(u)}{\partial u_\mu} = 0 \tag{28}$$

and the fact that for $t_x = t_y = t_z = t_{u'} = t_u$

$$[\psi^\tau(x),\, A^\tau_\mu(u)] = [\varphi^\tau(y),\, A^\tau_\mu(u)] = [\chi^\tau(z),\, A^\tau_\mu(u)]$$

$$= [A^\tau_\nu(u'),\, A^\tau_\mu(u)] = 0. \tag{29}$$

Calculating the divergence of both sides of Eq. (17) and using Eq. (27), we get general identities in an arbitrary gauge:

$$\frac{\partial}{\partial u^r_{\mu_r}} G_{\mu_1\ldots\mu_r\ldots\mu_m} = ie\frac{\partial\rho_{\mu_r}}{\partial u^r_{\mu_r}} G_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}. \tag{30}$$

In the special case of the Feynman gauge[5] they take the form

$$\frac{\partial}{\partial u^r_{\mu_r}} G^F_{\mu_1\ldots\mu_r\ldots\mu_m} = -e\sum_{l=1}^{n}[D_c(u^r - x^l)$$

$$- D_c(u^r - y^l)]\, G^F_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m},$$

$$D_c(x) = \frac{1}{(2\pi)^4}\int e^{ikx}\frac{dk}{k^2 - i\varepsilon}, \tag{31}$$

since the change from the L.K. gauge to the Feynman gauge is accomplished by means of a $\Lambda(x)$ for which

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*This is a purely illustrative example, since after such a transformation the commutator of the electromagnetic-field operators is no longer a c number.

$$\langle T\Lambda(x)\Lambda(y)\rangle = \frac{-i}{(2\pi)^4}\int e^{ik(x-y)}\frac{dk}{(k^2)^2}. \qquad (32)$$

If we now apply to Eq. (31) the d'Alembertian operator $\Box_{u^r}$, we get

$$\Box_{u^r}\frac{\partial}{\partial u^r_{\mu_r}} G^F_{\mu_1\ldots\mu_r\ldots\mu_m} = e\sum_{l=1}^{n}[\delta(u^r - x^l)$$
$$-\delta(u^r - y^l)]\, G^F_{\mu_1\ldots\mu_{r-1}\mu_{r+1}\ldots\mu_m}. \qquad (33)$$

With suitable definitions of the higher vertex parts the identities (30) lead to gauge-independent general Ward identities for processes with arbitrary numbers of charged and neutral particles and of photons. This derivation of the Ward identities is based only on the laws of gauge transformation of the fields, and assumes absolutely no knowledge of the concrete forms of the interactions or of their renormalizability.

As an example let us consider the case of a single charged particle $[\Psi = \psi(x)\overline{\psi}(y)]$ and m photons. Taking as the definition of the vertex part $\Gamma_{\mu_1\mu_2\ldots\mu_m}$ the formula

$$G_{\mu_1\mu_2\ldots\mu_m}(x, y, u^1 \ldots u^m) = (2\pi)^{-4(m+1)}e^m$$
$$\times\int dp^1dp^2dq^1 \ldots dq^m \exp(ip^1x - ip^2y$$
$$+ i\Sigma q^l u^l)\,\delta\Big(p^1 - p^2 + \sum_{l=1}^{m} q^l\Big)$$
$$\times\prod_{j=1}^{m} D^c_{\mu_j\nu_j}(q^j)\, S^c(p^1)\,\Gamma_{\nu_1\ldots\nu_m}(p^1, p^2, q^1 \ldots q^m)\, S^c(p^2) \quad (34)$$

(S^c is the exact one-particle Green's function of the charged particle), we get from Eq. (30) after dividing through by $q_r^2 d_l(q_r^2)$ and by all of the $D^c_{\mu_j\nu_j}(q^j)$:

$$S^c(p^1)\, q^r_{\nu_r}\Gamma_{\nu_1\ldots\nu_r\ldots\nu_m}(p^1, p^2, q^1 \ldots q^m)\, S^c(p^2)$$
$$= S^c(p^1)\,\Gamma_{\nu_1\ldots\nu_{r-1}\nu_{r+1}\ldots\nu_m}$$
$$\times(p^1, p^2 - q^r, q^1 \ldots q^{r-1}q^{r+1} \ldots q^m)\, S^c(p^2 - q^r)$$
$$- S^c(p^1 + q^r)\,\Gamma_{\nu_1\ldots\nu_{r-1}\nu_{r+1}\ldots\nu_m}$$
$$\times(p^1 + q^r, p^2, q^1 \ldots q^{r-1}q^{r+1} \ldots q^m)\, S^c(p^2), \qquad (35)$$

where

$$p^1 - p^2 + \sum_{l=1}^{m} q^l = 0.$$

For m = 1 we can write this identity in the form

$$S^c(p^1)\, q_\nu\Gamma_\nu(p^1, p^1 + q, q)\, S^c(p^1 + q) = S^c(p^1) - S^c(p^1 + q). \qquad (35a)$$

Let us also consider the case of two charged particles and examine the connections between the processes

$$\pi^+ + p \rightarrow \pi^+ + p, \quad \pi^+ + p \rightarrow \pi^+ + p + \gamma.$$

Introducing the standard definitions

$$G(x^1x^2, y^1y^2) = (2\pi)^{-12}\int dp^1\, dp^2\, dp^3\, dp^4$$
$$\times\exp(ip^1x^1 + ip^2x^2 - ip^3y^1 - ip^4y^2)$$
$$\times\delta(p^1 + p^2 - p^3 - p^4)\, G(p^1, p^2, p^3, p^4),$$
$$G_\mu(x^1x^2, y^1y^2, u) = (2\pi)^{-16}\, e\int dp^1\, dp^2\, dp^3\, dp^4\, dq$$
$$\times\exp(ip^1x^1 + ip^2x^2 - ip^3y^1 - ip^4y^2$$
$$+ iqu)\delta(p^1 + p^2 + q - p^3 - p^4)\, D^c_{\mu\nu}(q)\, S^c(p^1)\, \Delta^c(p^2)$$
$$\times\Gamma_\nu(p^1p^2, p^3p^4q)\, S^c(p^3)\, \Delta^c(p^4), \qquad (36)$$

where $S^c(p)$ and $\Delta^c(p)$ are the exact one-particle Green's functions of the proton and the π meson, we arrive at the identity

$$S^c(p^1)\, \Delta^c(p^2)\, q_\mu\Gamma_\mu(p^1, p^2, p^3, p^4, q)\, S^c(p^3)\, \Delta^c(p^4)$$
$$= G(p^1, p^2, p^3, p^4 - q) + G(p^1, p^2, p^3 - q, p^4)$$
$$- G(p^1, p^2 + q, p^3, p^4) - G(p^1 + q, p^2, p^3, p^4), \quad (37)$$

where $p^1 + p^2 + q = p^3 + p^4$.

The identities (35), which are a direct generalization of Ward's identity, were found by Fradkin,[2] who obtained them as a consequence of the Schwinger system of equations for the Green's functions in electrodynamics, i.e., by the use of a concrete form of renormalizable interaction. Quite recently identities of the form (35) have been derived by Kazes[7] by means of perturbation theory.

The identity (33) in the Feynman gauge is directly related to an identity for the coefficient functions of perturbation theory obtained by Bogolyubov and Shirkov[8] in quantum electrodynamics.

The generalized Ward identity (35a), also formulated by Green,[9] has also been proved without perturbation theory by Takahashi[10] by means of the formula (33) in the Feynman gauge, which he derived for m = 1. A gauge-independent derivation of this identity in electrodynamics has been given by Okubo[4] by the method of Caianello, which is closely connected with perturbation theory. Finally, Evans, Feldman, and Matthews[5] have obtained this same identity as a consequence of a law for gauge transformations of the Green's functions[1]

$$\langle T\psi(x)\overline{\psi}(y)\rangle \text{ and } \langle T\psi(x)\overline{\Psi}(y)A_\mu(u)\rangle,$$

which does not require any reference to the concrete form of the interaction.

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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

DEPENDENCE OF THE ANGULAR DISTRIBUTION OF FISSION FRAGMENTS ON THE SPIN OF THE TARGET NUCLEUS

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A simple analytic expression for the angular distribution of fission fragments is derived in the case of small anisotropy. The effect of the initial nuclear spin is discussed.

In reference 1, a method was suggested for the calculation of the angular distribution of fission fragments and analytic expressions for the angular distribution were obtained for a target nucleus of a given spin. In the case of a Gaussian distribution of the probability of fission with the value of the projection K of the angular momentum J of the compound nucleus in the direction of fission

$$\alpha(K) = \exp(-K^2/2K_0^2) \tag{1}$$

the angular distribution of the fragments depends weakly on the initial spin; the spin-dependent terms in the angular distribution are of the order $J_0^2 l^2/K_0^4$ (where J_0 is the initial spin, l is the orbital angular momentum of the neutron), while the basic term dependent on the angle is of the order l^2/K_0^2. The dependence of the total probability of decay of the compound nucleus Γ_t on the value of the angular momentum of the nucleus also leads to corrections of the second order in the angular distribution. This effect also leads to an additional weak dependence of the anisotropy on the initial spin. In the case of a small anisotropy in the angular distribution and a weak dependence of Γ_t on J, one can obtain an analytic expression for the angular distribution in which the dependence of Γ_t on J is taken into account.

The overall width of the compound nucleus state is the sum of the radiation, neutron, and fission widths. The first is usually negligibly small, while the dependence of Γ_n and Γ_f on the angular momentum can be determined with the aid of statistical theory:

$$\Gamma_n(J) \approx \Gamma_n(0) \exp[-(\alpha_f - \alpha_i) J(J+1)], \tag{2}$$

$$\Gamma_f(J) \approx (2J+1)^{-1} \Gamma_f(0) \exp[(\alpha_i - \alpha_\perp^*) J(J+1)] \times \sum_{K=-J}^{J} \exp[-K^2/2K_0^2]. \tag{3}$$

In (2) and (3), $\alpha = \hbar^2/2\mathcal{J}T$, where $\mathcal{J}$ is the moment of inertia of the nucleus and T is the nuclear temperature. The subscript i indicates quantities referring to the initial compound nucleus and f, to the compound nucleus state after the emission of the neutron; the asterisk indicates quantities referring to the deformed "transition" nucleus. The constant $\mathcal{J}_\perp$ appearing in the expression $\alpha_\perp^* = \hbar^2/2\mathcal{J}_\perp^* T^*$ is the moment of inertia of the transition nucleus with respect to the axis of symmetry. In the remaining cases, the moment of inertia is equal to the moment of inertia of a spherical nucleus $\mathcal{J}_0 = 2AmR^2/5$.

Expanding $\Gamma_n(J)$ and $\Gamma_f(J)$ into a series, we obtain

$$\gamma_f(J) = \Gamma_f(J)/\Gamma_t(J) \approx \gamma_f^0 [1 + qJ(J+1) + \ldots], \tag{4}$$

where

$$q = \gamma_n^{(0)} (\alpha_f - \alpha_\perp^*) + \gamma_f^{(0)} ({}^1/{}_2 K_0^2), \quad \gamma_n^{(0)} = \Gamma_n(0)/\Gamma_f(0),$$

$$\gamma_f^{(0)} = \Gamma_f(0)/\Gamma_t(0).$$

This expression for $\gamma_f(J)$ should be inserted into the general expression for the angular distribution of the fragments, which, for fixed values of l, of the channel spin $S = J_0 \pm 1/2$, and of J has the form[1]

$$W_{lSJ}(\vartheta) = \frac{2J+1}{2(2J_0+1)} \sum_{m\mu K} (C_{S\mu lm}^{JK})^2 a_J(K) |Y_{lm}(\vartheta)|^2, \qquad a_J(K) = \gamma_f(J)\alpha(K). \tag{5}$$

The overall angular distribution of the fragments is given by the expression

$$\overline{W}(\vartheta) = \sum_{lSJ} \zeta_l W_{lSJ}(\vartheta) \Big/ \sum_{lSJ} (2l+1)\zeta_l, \tag{6}$$

where ζ_l is the neutron sticking coefficient, which is assumed to be independent of J and S. Inserting Eqs. (5) and (4) into (6), we sum over J, for which we use the following formulas for the summation of the Clebsch-Gordan coefficients:

$$\sum_{J=|l-S|}^{l+S} (C_{S\mu lm}^{JK})^2 = \begin{cases} 1, & |K-m| \leqslant S \\ 0, & |K-m| > S \end{cases}$$

and[1]

$$\sum_{J=|l-S|}^{l+S} J(J+1)(C^{JK}_{S\mu lm})^2 = \begin{cases} l(l+1)+S(S+1)+2m\mu, & |\mu|\leqslant S, \quad |m|\leqslant l \\ 0, & |\mu|>S, \quad |m|>l \end{cases}.$$

After summing over J, we obtain

$$W_{lS}(\vartheta)=\sum_J W_{lSJ}(\vartheta)=C\sum_{m=-l}^{l}|Y_{lm}(\vartheta)|^2$$

$$\times\sum_{K=m-S}^{m+S}\{1+q[l(l+1)+S(S+1)$$

$$2m(K-m)]\}\,\alpha(K), \tag{7}$$

where the constant C does not depend on l and S. For α(K), we use the expansion

$$\alpha(K)\approx 1-K^2(2K_0^2)^{-1}+\tfrac{1}{2}\eta K^4(2K_0^2)^{-2}+\ldots. \tag{8}$$

An arbitrary coefficient η is introduced into formula (8) in order to take into account the possibility that the distribution α(K) deviates from a Gaussian one at large values of K. The term with K^4 leads to corrections of the second order. The correction to the angular distribution, as a result of the dependence of γ_f on J is small — of the order $q/2K_0^2$ —, i.e., it is a small quantity of the second order [q and $(2K_0^2)^{-1}$ are of the same order]. Taking this into account, we obtain from (6), (7), and (8), after simple calculations:

$$A(\vartheta)=[\sigma_f(\vartheta)-\sigma_f(90^\circ)]/\sigma_f(90^\circ)=(l^2/4K_0^2)\{[1+(\overline{S^2}/6K_0^2)$$

$$+q(\overline{l^4}/\overline{l^2}-\overline{l^2}+\tfrac{4}{3}\overline{S^2})]\cos^2\vartheta+(\eta/2K_0^2)[\tfrac{3}{8}\overline{l^{4\prime}}(1-\sin^4\vartheta)$$

$$+\overline{l^2}(\overline{S^2}+{}^1/_2)\cos^2\vartheta]\}, \tag{9}$$

where

$$\overline{l^2}=\sum_l(2l+1)\zeta_l l(l+1)\Big/\sum_l(2l+1)\zeta_l \tag{10}$$

is the mean square angular momentum transferred to the nucleus by the neutron; $\overline{l^4}$ and $\overline{l^{4\prime}}$ are the analogous mean values of the quantities $l^2(l+1)^2$ and $l(l^2-1)(l+2)$, and $\overline{S^2}=(J_0+{}^1\!/_2)^2+{}^1\!/_2$. In (9), we have discarded third-order terms. In the first order in $\overline{l^2}/2K_0^2$, the angular distribution does not depend on the spin of the target nucleus. The change in the anisotropy of the angular distribution, when the quantity $\overline{S^2}$ is changed to $\Delta\overline{S^2}$ is

$$\Delta_S A(0^\circ)=\tfrac{1}{4}\,\overline{l^2}K_0^{-2}\{(1-3\eta)/6K_0^2+\tfrac{4}{3}q\}\Delta\overline{S^2}. \tag{11}$$

According to the experimental data,[2] there is a small systematic difference in the value of the anisotropy for the nuclei U^{235}, U^{233}, and Pu^{239}:

$$A_{U^{235}}(0^\circ)-A_{U^{233}}(0^\circ)\approx A_{U^{233}}(0^\circ)-A_{Pu^{239}}(0^\circ)\approx 0.02-0.03.$$

It is natural to interpret this as an effect of the initial spin. For the three nuclei, the values of $\Delta\overline{S^2}$ are equal to 7 and 8 units. For a theoretical estimate of $\Delta_S A(0^0)$, we set $T_f = T^* = T$ in (4). The quantity $1/\mathscr{J}_f - 1/\mathscr{J}_\perp$ is equal to $\mathscr{J}_0$ $(1.2z + 5.6z^2)$, where $\mathscr{J}_0$ is the moment of inertia for a spherical nucleus, and $z = 1 - (Z^2A)/(Z^2/A)_{cr}$.[3] For T = 0.3 Mev, $\gamma_n^{(0)} \approx \gamma_f^{(0)} \approx 0.5$, and a value of $\mathscr{J}_0$ corresponding to a rigid body, we obtain $q \approx 0.01$. The parameter $\overline{l^2}/4K_0^2$ is determined directly from the experimental value of the anisotropy, since in a first approximation it is simply the same value. For neutrons of energy 3 — 5 Mev, we have $\overline{l^2}/4K_0^2 \approx 0.12$, $K_0^2 = 12 - 15$.[2,4] From (10) we find that the experimental value of the difference $\Delta A(0^\circ)$ for $\eta = 1$ [i.e. for a Gaussian α(K)] would correspond to q four to five times the thermodynamical value, which is unlikely. Another possible effect is a cutoff in the distribution α(K), in comparison with the Gaussian one, for large K, which is natural to expect in view of the finite size of the nucleus (see also ref. 1). The coefficient η should then be less than unity or negative. Then $\eta \approx 0$ would correspond to the experimental value of $\Delta A(0^\circ)$ for the above-mentioned thermodynamical value of q.

A rough estimate of the highest possible value K_{max} can be obtained by equating the rotational energy of the nucleus

$$E_{rot}\approx\hbar^2K^2/2J_{eff}=(K^2/2K_0^2)\,T^*$$

and the excitation energy of the transition nucleus E*. In this way, we obtain

$$K_{max}\approx\{(E^*/T^*)\cdot 2K_0^2\}^{1/2}.$$

For E* ≈ 2 — 3 Mev, we have $K_{max} \approx nK$, where the factor n is several units. This value of K_{max} is in agreement with $\eta \approx 0$, since for $\eta \approx 0$ the distribution α(K) vanishes when $K \approx \sqrt{2K_0^2}$.

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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

A THEORY OF THE DISPERSION OF THE HIGH-FREQUENCY EXCITON CONDUCTIVITY OF A CRYSTAL

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We have applied the diagram technique developed in reference 5 to the evaluation of the high-frequency conductivity. We obtain exciton states with large radii which occur near the threshold for the direct transition, whatever the position of the band extrema. We determine the exciton absorption spectrum and show that the fine structure of the exciton absorption lines is connected with a Doppler shift.

1. CLASSIFICATION OF THE DIRECT TRANSITIONS

THE absorption spectrum connected with the excitation of large radius excitons is a set of lines near the red limit of the main absorption of the crystal. The structure of the exciton states is connected in an essential way with the character of the interband transition responsible for the main absorption. The character of the interband transition is in turn determined by the relative position of the energy surfaces of the valence band and the conduction band in the Brillouin zone. The transitions that occur without the participation of phonons (direct transitions) are usually considered for the case (case a) where the extrema of the valence band and of the conduction band lie in the same point in momentum space. A number of authors[1-3] have considered the exciton states accompanying such a direct transition.

In the case when the energy maximum of the valence band and the minimum of the conduction band are in different points of $\mathbf{p}$-space (case b), one usually only pays attention to the indirect transitions that involve the participation of a phonon.[1] The exciton spectrum accompanying a direct transition of this latter type is of some interest. The absorption coefficient connected with the main transition is an increasing function of the frequency. The initial section of this curve is determined in case b by the indirect transition. The absorption coefficient is relatively small in this section. It increases steeply near the frequency ω_0 at which the direct transition becomes possible. The exciton absorption spectrum due to the direct transition will lie in the region of the indirect transition. It is the purpose of the present paper to study exciton states of this latter type.

The direct transition proceeds in case b from a point on the slope of the energy surface of the valence band to a point on the slope of the energy surface of the conduction band in such a way that these points correspond to the same wave vector apart from the small wave vector of the light. It is clear that such a transition occurs first at a frequency ω_0 corresponding to the minimum value for the difference between the energies in the bands at the same electron wave vector. Because of the crystal symmetry there will be in $\mathbf{p}$-space several such completely identical threshold points for the direct transition $\mathbf{K}_1, \ldots, \mathbf{K}_i, \ldots$.

We shall show that an exciton is formed near each threshold point for a direct transition $\mathbf{K}_i$. These excitons are equivalent, and their spectra are the same if we neglect the wave vector of the light. Each of them, however, is formed by the absorption of light by an electron which has a non-vanishing average velocity $\mathbf{v}_i = \partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}\,|_{\mathbf{p}=\mathbf{K}_i}$. It is clear that such an electron perceives the light with a Doppler shift $\Delta\omega_i = \boldsymbol{\kappa}\cdot\mathbf{v}_i$ where $\boldsymbol{\kappa}$ is the wave vector of the light. Since the velocities of the electrons in the different points $\mathbf{K}_i$ are in different directions, the exciton absorption line will be split into several components corresponding to the excitons in the different points $\mathbf{K}_i$, while the position of these components will depend on the orientation of the wave vector of the light with respect to the crystal axes. This effect is specific for the exciton absorption spectrum.

2. THE DIAGRAM TECHNIQUE FOR THE EVALUATION OF THE HIGH-FREQUENCY ELECTRON CONDUCTIVITY OF THE CRYSTAL

We construct the solution of the problem by using the simple model of an electron gas located in a periodic field, and we assume that the interaction between the electrons is sufficiently weak so that the effective interaction energy can be assumed to be much less than the width of the forbidden band. A phenomenological approach of this kind is the usual one in the theory of the Mott excitons.[1-3]

We consider a system of electrons in the periodic field of the lattice which fills the valence band. The Hamiltonian H of the system is of the form

$$H = H_0 + U; \qquad H_0 = \sum_{j\mathbf{p}} \varepsilon_{j\mathbf{p}} a^{+}_{j\mathbf{p}} a_{j\mathbf{p}},$$

$$U = \sum U^{j_3\mathbf{p}_3,\, j_4\mathbf{p}_4}_{j_1\mathbf{p}_1,\, j_2\mathbf{p}_2} a^{+}_{j_3\mathbf{p}_3} a^{+}_{j_4\mathbf{p}_4} a_{j_2\mathbf{p}_2} a_{j_1\mathbf{p}_1},$$

$$U^{j_3\mathbf{p}_3,\, j_4\mathbf{p}_4}_{j_1\mathbf{p}_1,\, j_2\mathbf{p}_2} = \sum_{\gamma, m, n} \frac{4\pi e^2}{\varepsilon \gamma^2 V} \delta_{\gamma - \mathbf{p}_1,\, \mathbf{p} - \mathbf{b}_n} \delta_{\gamma,\, \mathbf{p}_4 - \mathbf{p}_2 - \mathbf{b}_m} \Gamma_{j_1 j_3}$$
$$\times (\mathbf{p}_1, \mathbf{p}_3, \gamma)\, \Gamma_{j_2 j_4}(\mathbf{p}_2, \mathbf{p}_4, -\gamma),$$

$$\Gamma_{j_1 j_3}(\mathbf{p}_1, \mathbf{p}_3, \gamma) = \int_{V_0} e^{i\gamma\mathbf{x}} \psi^{*}_{j_3\mathbf{p}_3}(\mathbf{x})\, \psi_{j_1\mathbf{p}_1}(\mathbf{x})\, d\mathbf{x}. \tag{1}$$

Here $a^{+}_{j\mathbf{p}}$ and $a_{j\mathbf{p}}$ are respectively the creation and the annihilation operators of an electron in a state in the j-th band with a quasi-momentum **p**, which is described by a Bloch wave function $\psi_{j\mathbf{p}}(x)$; $\epsilon_{j\mathbf{p}}$ is the energy of this state; V is the normalizing volume; V_0 the volume of the elementary cell in the crystal; $\mathbf{b}_m$ and $\mathbf{b}_n$ are vectors which are multiples of the reciprocal lattice vector; the summation in the equation for U is over all the encountered band numbers and quasi-momenta. The quantity ϵ is the dielectric constant, which decreases the electron-electron interaction; we assume it to be a scalar for the sake of simplicity, although one can easily generalize the final result to the case where ϵ is a tensor.

We use Eq. (4a) of reference 5 to find a correction to the single-particle density matrix $f_{j\mathbf{p},j'\mathbf{p}'}$ which is proportional to the intensity of the applied electrical field $E_\mu(\mathbf{x}, t) = E_\mu(\boldsymbol{\kappa}, s)\exp(i\boldsymbol{\kappa}\cdot\mathbf{x} + st)$, where $s = -i\omega + \nu$, $\boldsymbol{\kappa}$ and ω are the wave vector and the frequency of the light, and ν is an adiabatic parameter (in the final equations $\nu \to 0$). We have, indeed,

$$f_{j\mathbf{p},\, j'\mathbf{p}'}(t)$$
$$= E_\mu(\mathbf{x}, s)\, e^{st} \sum_{i\mathbf{k},\, i'\mathbf{k}'} G^{i'\mathbf{k}',\, j'\mathbf{p}'}_{i\mathbf{k},\, j\mathbf{p}}(s, \beta) \int e^{i\kappa\mathbf{x}}\, j_\mu(\mathbf{x})_{i\mathbf{k},\, i'\mathbf{k}'} d\mathbf{x}; \tag{2a}$$

$$G^{i'\mathbf{k}',\, j'\mathbf{p}'}_{i\mathbf{k},\, j\mathbf{p}}(s, \beta) = Z^{-1} \int_{-\infty}^{0} e^{s\tau}\, d\tau \int_{0}^{\beta} d\lambda\, \mathrm{Sp}\left\{e^{-\beta H} \exp\left[\frac{H(\tau + i\hbar\lambda)}{i\hbar}\right] a^{+}_{j\mathbf{p}} a_{j'\mathbf{p}'}\right.$$
$$\left. \times \exp\left[-\frac{H(\tau + i\hbar\lambda)}{i\hbar}\right] a^{+}_{i\mathbf{k}} a_{i'\mathbf{k}'}\right\}, \tag{2b}$$

$$(j_\mu(\mathbf{x}))_{i\mathbf{k},\, i'\mathbf{k}'}$$
$$= (e/2m)\, [\psi^{*}_{i\mathbf{k}}(\mathbf{x})\, \hat{\mathbf{P}}_\mu \psi_{i'\mathbf{k}'}(\mathbf{x}) - \psi_{i'\mathbf{k}'}(\mathbf{x})\, \hat{\mathbf{P}}_\mu \psi^{*}_{i\mathbf{k}}(\mathbf{x})], \tag{2c}$$

$Z = \mathrm{Sp}\, e^{-\beta H}$, $\beta = T^{-1}$, T is the absolute temperature in energy units.

To evaluate the function $G^{i'\mathbf{k}',j'\mathbf{p}'}_{i\mathbf{k},j\mathbf{p}}(s, \beta)$ we use a diagram technique.[5] The feature of our case is that we have chosen the Bloch wave functions as the single-electron states. It is clear that in that case each line corresponds to the combination of the indices j and **p**. The vertex describing the electron-electron interaction is drawn in Fig. 1; it corresponds to the factor

$$\pm (i\hbar)^{-1} \sum_{mn} \frac{4\pi e^2 \varepsilon^{-1}}{V(\mathbf{p}_3 - \mathbf{p}_1 + \mathbf{b}_n)^2} \Gamma_{j_1 j_3}(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_3 - \mathbf{p}_1 + \mathbf{b}n)$$
$$\times \Gamma_{j_2 j_4}(\mathbf{p}_2, \mathbf{p}_4, \mathbf{p}_2 - \mathbf{p}_4 - \mathbf{b}_m)\, \delta_{\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4,\, \mathbf{b}_m - \mathbf{b}_n}.$$

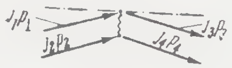

FIG. 1

The absolute magnitude of the quasi-momentum of any line is less than half a reciprocal lattice vector. Because of this, the quantity $\mathbf{b}_n - \mathbf{b}_m$ is less than a reciprocal lattice vector. It is clear that a factor $1 - n_{j\mathbf{p}}$ or $n_{j\mathbf{p}}$, where $n_{j\mathbf{p}}$ is a Fermi function, will correspond to a line. We shall consider the case $T \ll \hbar\omega_0'$; then $n_{j\mathbf{p}} = 0$, if j is an index of an unfilled band, and $n_{j\mathbf{p}} = 1$ if j is the index of a filled band.

We consider successively different approximations of perturbation theory. In Fig. 2 we give the zeroth order term. This diagram corresponds to the expression

$$\delta_{ij'}, \delta_{i'j} \delta_{\mathbf{pk}'} \delta_{\mathbf{kp}'} \sigma^{-1}(\sigma + \varepsilon_{j'\mathbf{p}'} - \varepsilon_{j\mathbf{p}})^{-1} [s + i\hbar^{-1}(\varepsilon_{j\mathbf{p}} - \varepsilon_{j'\mathbf{p}'})]^{-1}.$$

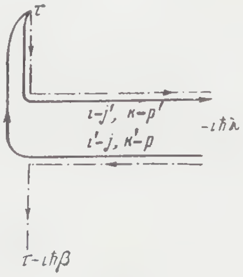

FIG. 2

The index j refers to a filled band and the index j′ to an empty band.

Since the frequency of the light is close to the threshold frequency ω_0 for a direct transition, the dominant zeroth order term will be the one where the index j is the same as the index v, referring to the valence band and when the index j′ is the same as the index c referring to the conduction band. For that term the denominator is a minimum. The threshold frequency for a direct transition is given by $\omega_0 = \mathrm{Min}\,(\epsilon_{\mathbf{c}\mathbf{p}+\kappa} - \epsilon_{\mathbf{v}\mathbf{p}})$. From the condition

$$\frac{\partial}{\partial p_\nu}(\varepsilon_{c\mathbf{p}+\varkappa} - \varepsilon_{v\mathbf{p}})\Big|_{\mathbf{p}=\mathbf{K}_i(\varkappa)} = 0$$

we can determine the momentum $\mathbf{K}_\mathrm{i}(\kappa)$ of the electrons which undergo a direct transition at the absorption threshold. It will be clear that only the two above-mentioned bands will be important. We introduce the notation

$$\int e^{i\varkappa\mathbf{x}}\,(j_\mu(\mathbf{x}))_{c\mathbf{k},\,v\mathbf{k}'}d\mathbf{x} = \delta_{\mathbf{k},\,\mathbf{k}'+\varkappa}\,j_\mu(\mathbf{k}',\varkappa),$$

$$j_\mu(\mathbf{k}',\varkappa) \equiv \int\limits_{V_0} e^{i\varkappa\mathbf{x}}\,(j_\mu(\mathbf{x}))_{c\mathbf{k}'+\varkappa,\,v\mathbf{k}'}d\mathbf{x}. \qquad (3)$$

One sees easily that

$$\int e^{-i\xi\mathbf{x}}(j_\nu(\mathbf{x}))_{v\mathbf{p},\,c\mathbf{p}'}\,d\mathbf{x} = \delta_{\mathbf{p}',\,\mathbf{p}+\xi}\;j^*_\nu(\mathbf{p},\xi).$$

We have not considered here the possibility of an Umklapp process for which $\mathbf{k}' + \kappa - \mathbf{k}' = \mathbf{b}$ ($\mathbf{b}$ is the reciprocal lattice vector), since the limits between which κ can vary are limited to a very small region of order κ near the edge of the Brillouin zone and the corresponding integral over $\mathbf{k}'$ will be small.

3. ELIMINATION OF SINGULAR DIAGRAMS

Some first-order diagrams are drawn in Fig. 3. We note a characteristic singularity of the diagram 3a. To do this we write down the contribution $J_\nu^{(1)}(\mathbf{y})$ to the average current density of the system in the point y:

$$J_\nu^{(1)}(\mathbf{y}) = \sum_{\mathbf{p}\mathbf{p}'} f^{(1)}_{v\mathbf{p},\,c\mathbf{p}'}\,(j_\nu(\mathbf{y}))_{v\mathbf{p},\,c\mathbf{p}'}. \qquad (4a)$$

Here

$$j_\nu(\mathbf{y})_{v\mathbf{p},\,c\mathbf{p}'} = \frac{1}{V}\sum_\xi e^{i\xi\mathbf{y}} j^*_\nu(\mathbf{p},\xi)\,\delta_{\mathbf{p}',\,\mathbf{p}+\xi}, \qquad (4b)$$

and $j_\nu(\mathbf{p},\xi)$ is defined by Eq. (3b);

$$f^{(1)}_{v\mathbf{p},\,c\mathbf{p}'} = -(i\hbar)^{-1}\sum_{\mathbf{k}\mathbf{k}'}\sum_{mn}\frac{E_\mu(\varkappa,s)\,e^{st}}{\varepsilon_{c\mathbf{k}}-\varepsilon_{v\mathbf{k}'}}\,\frac{\delta_{\mathbf{k},\,\mathbf{k}'+\varkappa}}{s+i(\varepsilon_{c\mathbf{k}}-\varepsilon_{v\mathbf{k}'})/\hbar}$$
$$\times\left[s+\frac{i}{\hbar}(\varepsilon_{c\mathbf{p}'}-\varepsilon_{v\mathbf{p}})\right]^{-1}\frac{4\pi e^2\varepsilon^{-1}}{V(\mathbf{k}'-\mathbf{p}+\mathbf{b}_n)^2}\,\Gamma_{vv}$$
$$\times(\mathbf{p},\mathbf{k}',\mathbf{k}'-\mathbf{p}+\mathbf{b}_n)\,\Gamma_{cc}(\mathbf{k},\mathbf{p}',\mathbf{k}-\mathbf{p}'-\mathbf{b}_m)\,\delta_{\varkappa-\xi,\,\mathbf{b}_m-\mathbf{b}_n}. \qquad (4c)$$

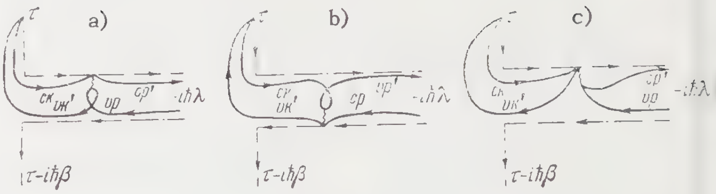

FIG. 3

The most important term in Eq. (4a) is, when we take (4b) and (4c) into account, the term with $\mathbf{b}_\mathrm{m} = \mathbf{b}_\mathrm{n} = 0$, in which there occurs an integral of the type

$$\int\frac{d\mathbf{k}'\,d\mathbf{p}}{(\mathbf{k}'-\mathbf{p})^2}A(\mathbf{k}',\mathbf{p})\left[s+\frac{i}{\hbar}(\varepsilon_{c\mathbf{k}'+\varkappa}-\varepsilon_{v\mathbf{k}'})\right]^{-1}$$
$$\times\left[s+\frac{i}{\hbar}(\varepsilon_{c\mathbf{p}+\varkappa}-\varepsilon_{v\mathbf{p}})\right]^{-1}.$$

The denominator in the integrand has at $\omega = \omega_0$ the form

$$s + i\hbar^{-1}(\varepsilon_{c\mathbf{p}+\varkappa}-\varepsilon_{v\mathbf{p}}) = \nu - \mu_{\alpha\beta}\,[\mathbf{p}-\mathbf{K}_i(\varkappa)]_\alpha\,[\mathbf{p}-\mathbf{K}_i(\varkappa)]_\beta$$

if $\mathbf{p}$ is near $\mathbf{K}_\mathrm{i}(\kappa)$; $A(\mathbf{k}',\mathbf{p})$ is a function which is finite for $\mathbf{k}' = \mathbf{p} = \mathbf{K}_\mathrm{i}(\kappa)$. It is clear that the integral diverges logarithmically as $\nu \to 0$. The terms connected with Umklapp processes for which $\mathbf{b}_\mathrm{m}$ or $\mathbf{b}_\mathrm{n}$ are different from zero are finite and small, since we have assumed that the reciprocal of the dielectric constant is sufficiently small. We shall therefore not consider Umklapp processes in the following, and put $\mathbf{b}_\mathrm{m} = \mathbf{b}_\mathrm{n} = 0$ at each vertex. Similarly we shall not take into account the diagrams of Figs. 3b and 3c. In these diagrams there is no divergence near the points $\mathbf{K}_\mathrm{i}(\kappa)$: in diagram 3b the denominator corresponding to the last section on the right does not tend to zero, while in diagram 3c the interaction matrix element does not tend to infinity.

Turning to higher-order approximations in the perturbation theory, we shall retain only those terms which have the largest singularity for a given power of the small parameter ϵ^{-1}. Such terms are drawn in Fig. 4a. Other terms, even though they diverge for $\epsilon^{-1} \ll 1$, can be neglected in comparison with the ones taken into account. One can, for instance, neglect the diagram of Fig. 5 in comparison with the diagram 4a, since it has an extra small factor ϵ^{-2} for the same degree of divergence.

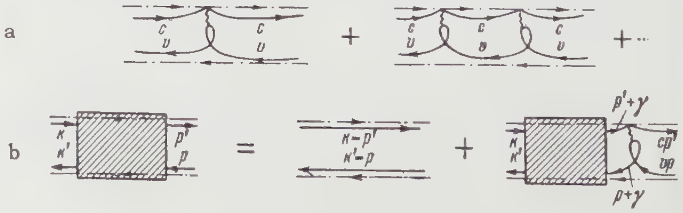

FIG. 4

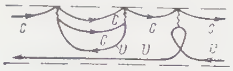

FIG. 5

All diagrams of Fig. 4a have a common factor corresponding to the horizontal section on the left so that it is convenient to introduce instead of the sum of the diagrams $G_{\mathbf{kp}}^{\mathbf{k'p'}}(s,\sigma)$ a quantity $F_{\mathbf{kp}}^{\mathbf{k'p'}}(s)$ such that

$$G_{\mathbf{kp}}^{\mathbf{k'p'}}(s,\sigma)=\frac{1}{\sigma}\frac{\delta_{\mathbf{k+p},\,\mathbf{k'+p'}}}{\sigma+\varepsilon_{c\mathbf{k}}-\varepsilon_{v\mathbf{k'}}}F_{\mathbf{kp}}^{\mathbf{k'p'}}(s). \tag{5}$$

We write down for the function $F_{\mathbf{kp}}^{\mathbf{k'p'}}(s)$ the equation depicted in Fig. 4b

$$\begin{aligned}F_{\mathbf{kp}}^{\mathbf{k'p'}}(s)&=\delta_{\mathbf{k'p}}\left[s+i\hbar^{-1}(\varepsilon_{c\mathbf{p'}}-\varepsilon_{v\mathbf{p}})\right]^{-1}\\&-(i\hbar)^{-1}\sum_{\gamma}F_{\mathbf{k},\,\mathbf{p}+\gamma}^{\mathbf{k'},\,\mathbf{p'}+\gamma}(s)\frac{4\pi e^2\varepsilon^{-1}}{V\gamma^2}\Gamma_{cc}(\mathbf{p}+\gamma,\,\mathbf{p},\,\gamma)\\&\times\Gamma_{vv}(\mathbf{p'},\,\mathbf{p'}+\gamma,\,-\gamma)\left[s+\frac{i}{\hbar}(\varepsilon_{c\mathbf{p'}}-\varepsilon_{v\mathbf{p}})\right]^{-1}.\end{aligned} \tag{6}$$

One should note that the interaction matrix element has in that equation the opposite sign, because of the intersection on the line v. The potential energy term in Eq. (6) describes therefore an attraction. When $\epsilon^{-1}\ll 1$ only small momentum transfers γ are important in Eq. (6).

The quantities Γ_{cc} and Γ_{vv} are equal to unity for small γ. If we take this into account and also the fact that $\mathbf{p'}=\mathbf{p}+\boldsymbol{\kappa}$ and $\mathbf{k}=\mathbf{k'}+\boldsymbol{\kappa}$ we can write Eq. (6) in the form

$$\begin{aligned}&(-i\hbar s+\varepsilon_{c\mathbf{p}+\varkappa}-\varepsilon_{v\mathbf{p}})\,R(\mathbf{p},\,\mathbf{k'})\\&\quad-\frac{4\pi e^2}{V}\sum_{\gamma}\frac{\varepsilon^{-1}}{\gamma^2}R(\mathbf{p}+\gamma,\,\mathbf{k'})=-i\hbar\delta_{\mathbf{p},\,\mathbf{k'}},\end{aligned} \tag{7}$$

$$R(\mathbf{p},\,\mathbf{k'})\equiv F_{\mathbf{k'}+\varkappa,\,\mathbf{p}}^{\mathbf{k'},\,\mathbf{p}+\varkappa}. \tag{8}$$

We have already pointed out that the difference $\epsilon_{c\mathbf{p}+\boldsymbol{\kappa}}-\epsilon_{v\mathbf{p}}$ is a minimum in a few points $\mathbf{K}_i(\boldsymbol{\kappa})$. The function $R(\mathbf{p},\mathbf{k'})$ can be seen to be large for large radius excitons only when $\mathbf{p}$ and $\mathbf{k'}$ are little different from $\mathbf{K}_i(\boldsymbol{\kappa})$. We shall find the function $R(\mathbf{p},\mathbf{k'})$ in the neighborhood of the point $\mathbf{K}_i(\boldsymbol{\kappa})\equiv\mathbf{K}_i+\boldsymbol{\lambda}$, $\mathbf{K}_i\equiv\mathbf{K}_i(0)$. In that region Eq. (7) is of the form

$$\begin{aligned}&\Big\{-i\hbar s+\varepsilon_{c\mathbf{K}_i+\boldsymbol{\lambda}+\varkappa}-\varepsilon_{v\mathbf{K}_i+\boldsymbol{\lambda}}+\frac{1}{2}\left[\frac{\partial^2}{\partial p_\nu\,\partial p_\mu}(\varepsilon_{c\mathbf{p}+\varkappa}-\varepsilon_{v\mathbf{p}})\right]_{\mathbf{p}=\mathbf{K}_i+\boldsymbol{\lambda}}\\&\quad\times(p-K_i-\lambda)_\nu(p-K_i-\lambda)_\mu\Big\}R(\mathbf{p},\,\mathbf{k'})\\&\quad-\sum_{\gamma}\frac{4\pi e^2\varepsilon^{-1}}{\gamma^2V}R(\mathbf{p}+\gamma,\,\mathbf{k'})=-i\hbar\delta_{\mathbf{p},\,\mathbf{k'}}.\end{aligned} \tag{9}$$

We use the fact that $\boldsymbol{\kappa}$ and the quantity λ which is proportional to $\boldsymbol{\kappa}$ are small, and expand the kinetic energy (the sum within the square brackets) in Eq. (9), retaining no terms beyond those quadratic in $\boldsymbol{\kappa}$ and $\mathbf{p}-\mathbf{K}_i$. Equation (9) then becomes

$$\begin{aligned}&\{-i\hbar s+E_0+\hbar\mathbf{v}^i\varkappa+\tfrac{1}{2}M_{\alpha\beta}^{-1}\hbar^2\varkappa_\alpha\varkappa_\beta+\tfrac{1}{2}\mu_{\alpha\beta}^{-1}\hbar^2(p-K_i-\lambda)_\alpha\\&\quad\times(p-K_i-\lambda)_\beta\}\,R(\mathbf{p},\,\mathbf{k'})\\&\quad-\sum_{\gamma}\frac{4\pi e^2\varepsilon^{-1}}{\gamma^2V}R(\mathbf{p}+\gamma,\,\mathbf{k'})=-i\hbar\delta_{\mathbf{p},\,\mathbf{k'}},\end{aligned} \tag{10}$$

$$E_0=\hbar\omega_0,\ \lambda_\alpha=-[(m_c^{-1}+m_v^{-1})^{-1}m_c^{-1}]_{\alpha,\,\beta}\varkappa_\beta,$$

$$M_{\alpha\beta}^{-1}=[m_c^{-1}(m_c^{-1}+m_v^{-1})^{-1}m_v^{-1}]_{\alpha\beta},\quad\mu_{\alpha\beta}^{-1}=(m_c^{-1}+m_v^{-1})_{\alpha\beta}, \tag{11}$$

where

$$(m_c^{-1})_{\alpha\beta}=\frac{1}{\hbar^2}\frac{\partial^2\varepsilon_{c\mathbf{p}}}{\partial p_\alpha\,\partial p_\beta}\bigg|_{\mathbf{p}=\mathbf{K}_i},\quad(m_v^{-1})_{\alpha\beta}=-\frac{1}{\hbar^2}\frac{\partial^2\varepsilon_{v\mathbf{p}}}{\partial p_\alpha\,\partial p_\beta}\bigg|_{\mathbf{p}=\mathbf{K}}$$

$$v_\alpha^i=\frac{1}{\hbar}\frac{\partial\varepsilon_{c\mathbf{p}}}{\partial p_\alpha}\bigg|_{\mathbf{p}=\mathbf{K}_i}=\frac{1}{\hbar}\frac{\partial\varepsilon_{v\mathbf{p}}}{\partial p_\alpha}\bigg|_{\mathbf{p}=\mathbf{K}_i}. \tag{12}$$

One checks easily that the solution of Eq. (10) is of the form

$$R(\mathbf{p},\,\mathbf{k'})=\begin{cases}\dfrac{1}{V}\displaystyle\sum_n\frac{\varphi_n(\mathbf{p}-\mathbf{K}_1-\boldsymbol{\lambda})\,\varphi_n^*(\mathbf{k'}-\mathbf{K}_1-\boldsymbol{\lambda})}{s+i(\omega_n+\mathbf{v}^i\varkappa)}, & \mathbf{p},\ \mathbf{k'}\sim\mathbf{K}_1+\boldsymbol{\lambda},\\[2ex]\dfrac{1}{V}\displaystyle\sum_n\frac{\varphi_n(\mathbf{p}-\mathbf{K}_i-\boldsymbol{\lambda})\,\varphi_n^*(\mathbf{k'}-\mathbf{K}_i-\boldsymbol{\lambda})}{s+i(\omega_n+\mathbf{v}^i\varkappa)}, & \mathbf{p},\ \mathbf{k'}\sim\mathbf{K}_i+\boldsymbol{\lambda}.\end{cases} \tag{13}$$

In all other cases R vanishes. In (13) $\hbar\omega_n=E_0+\frac{1}{2}M_{\alpha\beta}^{-1}\hbar^2\kappa_\alpha\kappa_\beta+\epsilon_n$ and the wave function $\varphi_n(\mathbf{q})$ of the n-th exciton state satisfies the equation

$$\mu_{\alpha\beta}^{-1}\frac{\hbar^2}{2}q_\alpha q_\beta\varphi_n(\mathbf{q})-\sum_{\gamma}\frac{4\pi e^2}{\gamma^2}\frac{\varepsilon^{-1}}{V}\varphi_n(\mathbf{q}+\gamma)=\varepsilon_n\varphi_n(\mathbf{q}) \tag{14}$$

with eigenvalue ϵ_n.

If we take the Fourier transform of Eq. (14) we arrive at the normal Schrödinger equation, in coordinate representation, describing the internal motion of the exciton. $\varphi_n(\mathbf{q})$ turns out then to be the Fourier component of the normalized wave function $\tilde{\varphi}_n(\mathbf{r})$ in coordinate representation:

$$\varphi_n(\mathbf{q})=\int e^{i\mathbf{qr}}\tilde{\varphi}_n(\mathbf{r})\,dr,\quad\int|\tilde{\varphi}_n(\mathbf{r})|^2\,dr=1. \tag{15}$$

4. THE HIGH-FREQUENCY CONDUCTIVITY AND THE ABSORPTION COEFFICIENT

Using Eqs. (13), (8), (5), (2a) and equations similar to (4a) and (4b), we find the high-frequency conductivity $\sigma_{\nu\mu}(\boldsymbol{\kappa},\omega)$

$$\begin{aligned}\sigma_{\nu\mu}(\varkappa,\,\omega)&=E_0^{-1}(2\pi)^{-6}\\&\times\sum_{n,i}\int d\mathbf{p}\,d\mathbf{k'}\,\frac{\varphi_n(\mathbf{p}-\mathbf{K}_i-\boldsymbol{\lambda})\,\varphi_n^*(\mathbf{k'}-\mathbf{K}_i-\boldsymbol{\lambda})}{s+i(\omega_n+\mathbf{v}^i\varkappa)}j_\mu(\mathbf{k'},\,\varkappa)j_\nu^*(\mathbf{p},\,\varkappa).\end{aligned} \tag{16}$$

For allowed transitions, when $j_\mu(\mathbf{K}_i, 0) \neq 0$, the expression for $\sigma_{\nu\mu}$ can be simplified, since $\varphi(\mathbf{p} - \mathbf{K}_i - \boldsymbol{\lambda})$ is different from zero for large radius excitons only in a small region of $\mathbf{p}$-space near the point $\mathbf{K}_i + \boldsymbol{\lambda}$. One can then neglect the $\mathbf{k}'$ dependence of the quantity $j_\mu(\mathbf{k}', \boldsymbol{\kappa})$, and remove it from under the integral sign at the point $\mathbf{K}_i$. The exciton high-frequency polarizability $\chi_{\nu\mu}(\boldsymbol{\kappa}, \omega) = \sigma_{\nu\mu}(\boldsymbol{\kappa}, \omega)/i\omega$ is then of the form

$$\chi_{\nu\mu}(\boldsymbol{\varkappa}, \omega) = -\frac{e^2}{\hbar} \sum_n |\widetilde{\varphi}_n(0)|^2 \sum_j \frac{r_{vc}^{\nu}(\mathbf{K}_j)\, r_{cv}^{\mu}(\mathbf{K}_j)}{(\omega - \omega_n - \mathbf{v}^j \boldsymbol{\varkappa}) + i\nu},$$

$$r_{vc}^{\nu}(\mathbf{K}_j) = \int_{V_0} \psi_{v\mathbf{K}_j}^{*}(\mathbf{r})\, r^{\nu}\, \psi_{c\mathbf{K}_j}(\mathbf{r})\, d\mathbf{r}. \tag{17}$$

This expression has the same form as the equation for the polarizability of a gas, with this difference that the quantity $|\widetilde{\varphi}_n(0)|^2$ plays the part of the concentration of the gas, so that the average distance between the "gas molecules" is equal to the exciton radius.

If we take relaxation processes into account we are led to a finite value of ν in Eq. (17). ν is then the reciprocal of the relaxation time. One obtains easily from Eq. (17) in that case an estimate for an expression for the exciton polarizability χ_l in one of the exciton absorption lines

$$\chi_l \sim (\omega_0/\nu)\,(r_a/r_e)^3, \tag{18}$$

where r_a is a quantity of the order of the interatomic distance and r_e the exciton radius.

If $\chi_l \geq (4\pi)^{-1}\epsilon$ one must solve a dispersion equation to find $\boldsymbol{\kappa}$ as a function of ω and take the exciton polarizability into account. The picture of the propagation of light through the crystal becomes then complicated (see references 6 and 7). In the present paper we restricted ourselves to the case $\chi_l \ll (4\pi)^{-1}\epsilon$. In that case $\kappa = (\omega/c)\sqrt{\epsilon}$. One obtains easily from Eq. (17) the absorption coefficient $\mathscr{K}_x(\boldsymbol{\kappa}, \omega)$ for an electromagnetic wave which is polarized along the x axis

$$\mathscr{K}_x(\boldsymbol{\varkappa}, \omega) = \frac{4\pi e^2}{\hbar c}\,\omega_0 \sum_n |\widetilde{\varphi}_n(0)|^2 \sum_j \frac{|r_{vc}^{x}|_j^2\, \nu}{\nu^2 + (\omega - \omega_n + \mathbf{v}^j \boldsymbol{\varkappa})^2}. \tag{19}$$

As before, we have assumed here that $\epsilon \gg 1$.

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RELATIVISTICALLY COVARIANT RELATIONS BETWEEN POLARIZATION EFFECTS IN THE SCATTERING OF SPIN ½ PARTICLES

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The relations between all quantities characterizing polarization effects in the scattering of different and identical particles of spin ½ are derived in a relativistically covariant form.

THE relations between polarization effects in the scattering of spin ½ particles were obtained in nonrelativistic form by Puzikov, Ryndin, and Smorodinskii[1] and Baz'.[2] It is of interest to obtain these relations for the relativistic case.

1. We consider the scattering of two different particles of momenta P and p, masses M and m, and polarization 4-vectors Z_μ and ξ_μ, respectively.

The state of polarization of a relativistic particle of spin ½ is described by the matrix of density[3]

$$\rho = \tfrac{1}{2}(1 + \sigma\xi)\,\eta^{(+)}.$$

Here $\sigma\xi = \sigma_\mu\xi_\mu$, $\sigma_\mu = i\gamma_5\gamma_\mu$; ξ_μ is the polarization 4-vector, and $\eta^{(+)}$ is the operator of the projection on a state of positive energy. The formulas for the different polarization effects can be written in the form

$$\sigma d/d\Omega \sim R = \operatorname{Sp} L,\ Z'_\nu R = \operatorname{Sp}\{\Sigma_\nu L\},$$
$$\xi'^\alpha R = \operatorname{Sp}\{\sigma^\alpha L\},\ Z'_\nu \xi'^\alpha R = \operatorname{Sp}\{\Sigma_\nu\sigma^\alpha L\}, \tag{1}$$

where

$$L = \Lambda'\lambda' A\Lambda(1 + \Sigma Z)\Lambda\lambda(1 + \sigma\xi)\lambda\bar{A}\lambda'\Lambda'.$$

Here A is determined by the expression for the matrix element

$$M_{if} = \overline{U}(P')\,\bar{u}(p')\,A\,u(p)\,U(P),$$
$$\bar{A} = \Gamma_4\gamma_4 A^+\gamma_4\Gamma_4,\ \Lambda = M - i\hat{P},\ \lambda = m - i\hat{p}.$$

The upper case letters and subscripts refer to the first particle, the lower case letters and superscripts refer to the second particle; the final state is indicated by the prime.

We rewrite formulas (1) in somewhat different form, so that the polarization coefficients are separated:

$$R = r + s_\mu Z_\mu + c^\beta\xi^\beta + d^\beta_\mu Z_\mu\xi^\beta,$$
$$Z'_\nu R = l_\nu + m_{\nu\mu} Z_\mu + n^\beta_\nu\xi^\beta + q^\beta_{\nu\mu} Z_\mu\xi^\beta,$$
$$\xi'^\alpha R = f^\alpha + g^\alpha_\mu Z_\mu + h^{\alpha\beta}\xi^\beta + t^{\alpha\beta}_\mu Z_\mu\xi^\beta,$$
$$Z'_\nu\xi'^\alpha R = u^\alpha_\nu + v^\alpha_{\nu\mu} Z_\mu + x^{\alpha\beta}_\nu\xi^\beta + y^{\alpha\beta}_{\nu\mu} Z_\mu\xi^\beta. \tag{2}$$

It is clear in which experiments these polarization coefficients are measured.

Since we have in mind interactions in which parity is conserved, we use, instead of the invariance of the scattering amplitude with respect to time reversal, the equivalent invariance with respect to charge conjugation:

$$A(p, p', P, P') = CcA^T(-p', -p, -P', -P)\,c^{-1}C^{-1}. \tag{3}$$

Here T denotes the transpose with respect to the spinor indices of both particles. Inserting (3) into the explicit expression for the polarization coefficients obtained from (1) and (2), we find the relations

$$l_\mu = \overset{\simeq}{s}_\mu,\quad c^\beta = \overset{\simeq}{f}{}^\beta,\quad d^\beta_\mu = \overset{\simeq}{u}{}^\beta_\mu,$$
$$g^\beta_\mu = \overset{\simeq}{n}{}^\beta_\mu,\quad q^\beta_{\nu\mu} = \overset{\simeq}{v}{}^\beta_{\mu\nu},\quad t^{\alpha\beta}_\mu = \overset{\simeq}{x}{}^{\beta\alpha}_\mu. \tag{4}$$

Moreover, we impose the following conditions on some of the coefficients:

$$m_{\mu\nu} = \overset{\simeq}{m}_{\nu\mu},\quad h^{\alpha\beta} = \overset{\simeq}{h}{}^{\beta\alpha},\quad y^{\alpha\beta}_{\nu\mu} = \overset{\simeq}{y}{}^{\beta\alpha}_{\mu\nu}. \tag{5}$$

The sign ≃ over the letter denotes the substitution

$$p \rightleftarrows -p',\ P \rightleftarrows -P'. \tag{6}$$

In particular, the result obtained earlier by Bilenkii and Ryndin[4] follows from formulas (4):

$$s_\mu = l_\mu,\quad c^\beta = f^\beta. \tag{7}$$

In fact, the most general form of the expression for each of these pseudovectors has the form $N_\mu F$, where

$$N_\mu = i\varepsilon_{\mu\nu\lambda\rho}(p + p')_\nu(P + P')_\lambda(p - p')_\rho,$$

and F is a function only of invariants. Since neither N_μ nor F change under the substitution (6), we obtain formula (7) from (4).

2. The interaction between particles can sometimes be of the type in which the scattering ampli-

tude is an invariant under charge conjugation with respect to only one of the particles, i.e.,

$$A(p,\ p',\ P,\ P') = \beta C A^t(p,\ p',\ -P',\ -P)\,C^{-1}. \qquad (8)$$

Here $|\beta^2| = -1$, t denotes the transpose with respect to the spinor indices of the first particle. A similar situation is encountered, for example, when we consider the scattering of electrons by protons in the first Born approximation for the electromagnetic field,[5] where $\beta = -1$.*

In this case, if we insert (8) in the formulas for the polarization coefficients, we find the following relations:

$$d_\mu^\beta = \widetilde{n}_\mu^\beta,\quad g_\mu^\beta = \widetilde{u}_\mu^\beta. \qquad (9)$$

Moreover, the demand (8) imposes a number of conditions on the coefficients:

$$r = \widetilde{r},\quad l_\mu = \overline{l}_\mu,\quad s_\mu = \overline{s}_\mu,\quad c^\beta = \widetilde{c}^\beta,$$
$$m_{\nu\mu} = \overline{m}_{\nu\mu},\quad q_{\nu\mu}^\beta = \widetilde{q}_{\mu\nu}^\beta,\quad f^\beta = \widetilde{f}^\beta,$$
$$h^{\alpha\beta} = \widetilde{h}^{\alpha\beta},\quad x_\mu^{\alpha\beta} = \overline{x}_\mu^{\beta\alpha},\quad t_\mu^{\alpha\beta} = \overline{t}_\mu^{\beta\alpha},\quad v_{\mu\nu}^\beta = \widetilde{v}_{\nu\mu}^\beta,\quad y_{\nu\mu}^{\alpha\beta} = \widetilde{y}_{\mu\nu}^{\alpha\beta}. \qquad (10)$$

Here the bar above the letter denotes the substitution p ⇄ − p′; the tilde denotes the substitution P ⇄ − P′.

Relations (9), however, bear a character different from (4). Since the invariants change when we make the substitution P ⇄ − P′, then we can find, say n_μ^β, only if the analytic expression for d_μ^β is known.

We note that the expression for the polarization effects in the scattering of electrons by protons[5,7] satisfies the demands (4), (5), (9), and (10).

3. If the particles are identical, then the scattering amplitude possesses definite symmetry properties, namely:

$$A_{\mu\nu}^{\alpha\beta}(p,\ p',\ P,\ P') = -A_{\alpha\nu}^{\mu\beta}(p,\ P',\ P,\ p'),$$
$$A_{\mu\nu}^{\alpha\beta}(p,\ p',\ P,\ P') = \quad A_{\alpha\beta}^{\mu\nu}(P,\ P',\ p,\ p').$$

This leads to the following auxiliary relations between the coefficients:

$$s_\mu = \underset{\approx}{c}^{\mu},\quad l_\mu = \underset{\approx}{f}^{\mu},\quad m_{\nu\mu} = \underset{\mp}{h}^{\nu\mu},$$
$$n_\mu^\beta = \underset{\approx}{g}_\beta^\mu,\quad q_{\nu\mu}^\beta = \underset{\approx}{t}_\beta^{\nu\mu},\quad v_{\nu\mu}^\beta = \underset{\approx}{x}_\beta^{\nu\mu}; \qquad (11)$$

$$m_{\nu\mu} = \underset{\sim}{g}_\mu^\nu,\quad n_\mu^\beta = \underset{\sim}{h}^{\mu\beta}. \qquad (12)$$

Here ≈ under the letter denotes the substitution P ⇄ p and P′ ⇄ p′, and ~ denotes the substitution P′ ⇄ p′.

The following conditions are also imposed on the coefficients:

$$r = \underset{\sim}{r} = \underset{\approx}{r},\quad d_\mu^\beta = \underset{\sim}{d}_\mu^\beta = \underset{\approx}{d}_\beta^\mu,\quad s_\mu = \underset{\sim}{s}_\mu,$$
$$u_\mu^\beta = \underline{u}_\mu^\beta = \underset{\approx}{u}_\beta^\mu,\quad y_{\nu\mu}^{\alpha\beta} = \underset{\sim}{y}_{\alpha\mu}^{\nu\beta} = \underset{\approx}{y}_{\alpha\beta}^{\nu\mu},$$
$$c^\beta = \underset{\sim}{c}^\beta,\quad f^\beta = \underline{f}^\beta,\quad l_\mu = \underline{l}_\mu,\quad q_{\nu\mu}^\beta = \underline{q}_{\nu\beta}^\mu,$$
$$t_\mu^{\alpha\beta} = \underline{t}_\beta^{\alpha\mu},\quad v_{\nu\mu}^\beta = \underset{\sim}{v}_{\beta\mu}^\nu,\quad x_\mu^{\alpha\beta} = \underset{\sim}{x}_\alpha^{\mu\beta}. \qquad (13)$$

Here the bar under the letter denotes the substitution P ⇄ p.

Some of the relations (11) are simply equalities:

$$s_\mu = c^\mu,\quad l_\mu = f^\mu.$$

The reasoning is the same as in the derivation of equalities (7).

What was said above concerning relations (9) applies to relations (12), since the invariants change when we make the substitution P′ ⇄ p′.

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*Edwards and Matthews[6] call such an invariance property for $\beta = 1$ crossing symmetry for fermions.

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PAIRING FORCES AND PAIR CORRELATIONS IN THE NUCLEI Tl^{206} *AND* Bi^{210}

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The residual pairing forces acting between a neutron and proton in the nucleus are determined. The energies, eigenfunctions and transition probabilities in the nuclei Tl^{206} and Bi^{210} are found and compared with experiment.

1. INTRODUCTION

THIS paper is devoted to the determination of the residual forces which act between a neutron and a proton in their motion in the self-consistent field of the nucleus. The most convenient nuclei for this purpose are those whose nucleon configurations differ from doubly closed shells by the presence of two holes or two "additional" nucleons. In the region of heavy nuclei, Tl^{206} and Bi^{210} are of this type.

At first glance it might seem that in a heavy nucleus the neutron and proton, being in different shells, interact weakly with one another. However one can convince oneself that this is not so by comparing the β decay energies of the nuclei Tl^{207} and Tl^{206} or the nuclei Pb^{209} and Pb^{210}. In the $Tl^{207} \rightarrow Pb^{207}$ β decay, an energy of 1400 kev is liberated. This energy is made up of the difference in the energy levels of the proton and neutron in the nuclei, and of the mass difference which is 780 kev. The β decay energy of Tl^{206} differs from that for Tl^{207} by the difference between the binding energies ϵ_{np} of a neutron-proton pair and ϵ_{nn} for a neutron-neutron pair. If we assume that the neutron-proton interaction is small, i.e., $\epsilon_{np} \ll \epsilon_{nn}$, then the β decay energies of Tl^{207} and Tl^{206} should differ by approximately 700 kev. The observed difference in β decay energy is only 70 kev. Consequently $\epsilon_{np} \approx \epsilon_{nn}$, and the neutron-proton interaction is of the same order as the neutron-neutron interaction. Similarly, from a consideration of the $Pb^{210} \rightarrow Bi^{210}$ β decay it follows that the binding of the n-p pair in Bi^{210} is also not less than 700 kev. For this reason an investigation of the n-p interaction in the Tl^{206} and Bi^{210} nuclei was made by the same method which was used for studying the n-n forces in Pb^{206}.[1]

The motion of the external nucleons is described by the potential

$$V = \sum_{i=1,2} V_C(i) + \sum_{i=1,2} V_S(i) + V_p(1, 2). \tag{1}$$

Here

$$V_C(i) = V(r_i) - \lambda \left(\frac{\hbar}{2Mc}\right)^2 \frac{\mathbf{l}_i \mathbf{s}_i}{r_i} \frac{\partial V(r_i)}{\partial r_i},$$

$$V(r_i) = \frac{V_0}{1 + \exp\{\alpha(r_i - r_0)\}} \qquad (2)^*$$

is the diffuse single particle central potential for the i-th external particle, whose parameters were determined in reference 2;

$$V_S(i) = -\varkappa(r_i) \sum_{\mu} \alpha_{2\mu} Y_{2\mu}(\vartheta_i, \varphi_i) \tag{3}$$

is the potential for the quadrupole interaction of the i-th external particle with the particles of the core;

$$V_p(1, 2) = -\exp\{-(\mathbf{r}_1 - \mathbf{r}_2)^2/\rho^2\}(v_t\pi_t + v_s\pi_s) \tag{4}$$

(in which π_s and π_t are the single and triplet projection operators) is the potential of the residual pairing interaction between neutron and proton. The parameter ρ corresponds to the effective radius of interaction of the nuclear forces, while v_s and v_t determine the depth of the singlet and triplet forces.[1]

Using the potential (1), we calculated the spectrum of excited states for the Tl^{206} and Bi^{210} nuclei, in order to determine the parameters of the pairing interaction.

2. SPECTRUM OF Tl^{206}

To calculate the spectrum of Tl^{206} we used the single-particle levels and the corresponding wave functions for the nuclei Pb^{207} and Tl^{207}. A good determination of the levels of Pb^{207} was made in reference 2. For Tl^{207}, the binding energy of the proton in the ground $s_{1/2}$ state is known. The location of the first excited $d_{3/2}$ level can be determined by tracing this level in all the Tl isotopes,

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$^*\mathbf{l}_i\mathbf{s}_i = \mathbf{l}_i \cdot \mathbf{s}_i$.

from which one sees that for Tl^{206} the $d_{3/2}$ level should lie at ~ 150 kev. The levels which follow are determined by solving the Schrödinger equation with the potential V_C in the form (2); in particular, the $d_{5/2}$ level turns out to be at a height of 1150 kev, while the other levels are higher.

The experimental spectrum for Tl^{206} has been studied in detail by Rusinov et al.[3] The spectrum is characterized by the occurrence of narrow doublets.

Turning to the theoretical interpretation of the spectrum of Tl^{206}, we should remark that if we omit the n-p interaction it is impossible to explain the sequence of levels, even forgetting about the binding energy of pairs. It is therefore necessary to solve the Schrödinger equation with the complete potential (1). The method of solution was described earlier.[1] Briefly, it reduces to the diagonalization of the energy matrices calculated for definite total angular momentum and parity. As basis functions we use linear combinations of products of single particle functions for the neutron and proton, and a wave function describing the quadrupole oscillations of the core,

$$|(l_1 j_1,\ l_2 j_2) J,\ NR;\ IM\rangle. \tag{5}$$

From the diagonalization of the energy matrix, we get the sequence of the energy levels of the nucleus which have a given spin and parity, and also the coefficients $c^I\ (l_1 j_1,\ l_2 j_2,\ J;\ NR)$ of the linear combinations of the basis functions (5), which determine the corresponding wave functions Ψ_{IM}:

$$\Psi_{IM} = \sum c^I (l_1 j_1,\ l_2 j_2,\ J;\ \ NR) |(l_1 j_1,\ l_2 j_2) J;\ NR;\ IM\rangle \tag{6}$$

[The summation in (6) extends over all variables except I and M.] The degree of the energy matrix is limited by the requirement that the energies and wave functions of the low-lying levels be determined to sufficient accuracy. Comparison of the results of the calculation with the experimental spectrum enables us to determine the optimum values of the parameters in the potential.

As it turns out, the doublets in the Tl^{206} spectrum are especially sensitive to the parameters v_t and v_s, which makes the determination of these parameters easier. For the parameters $\hbar\omega$ and C, on which the $\alpha_{2\mu}$ depend, and also for ρ, we took the values obtained from analysis of the Pb^{206} spectrum, since these parameters cannot be very different for neighboring nuclei.

In Table I we give the energy levels calculated for fixed values of the parameters v_t and v_s, omitting the interaction of the nucleons with the nuclear surface. From this table we see that there are three doublets in the Tl^{206} spectrum. As we see from Table II, variation of the parameters v_t and v_s, and including the interaction of the nucleons with the core, leads to different doublet separations and different splittings of the doublets themselves, but does not mix them. Thus there is no question that there is an excited level with $I = 1^-$ near the ground state.

Figure 1 shows the level scheme of Tl^{206} as obtained with the parameters $v_t = 22$ Mev and $v_s = 27$ Mev, values which were determined from the energy splitting of the second doublet (40 kev) and the binding energy of the pair in the ground state. For these same values of the parameters v_t and v_s, we calculated the wave functions of the doublets, i.e., the coefficients $c^I\ (l_1 j_1,\ l_2 j_2,\ J;\ NR)$, which are given in Table III. The theoretical level scheme is in quite good agreement with the experimental data, and one could obtain even better agreement with experiment, for example, by reducing the parameter ρ by 3 — 5%. But since the energies of the single particle levels are not known exactly, the calculation was not pursued to obtain better agreement with the experimental doublet separations.

Using these wave functions for the Tl^{206} levels, we calculated the reduced probabilities $B(\lambda)$ for γ transitions; the results are shown in Table V. For the electric transitions $2^- \rightarrow 0^-$ and $2^- \rightarrow 1^-$ the theoretical values of $B(E2)$ are in good agreement with the experimental data. From this it fol-

Table I. Energy levels of Tl^{206} (in Mev) calculated for $v_t = 20$ and $v_s = 20$, including only pairing interaction

Number of level	$I = 0^-$		$I = 1^-$		$I = 2^-$		$I = 3^-$	
	energy of single-particle levels	final energy of level	energy of single-particle levels	final energy of level	energy of single-particle levels	final energy of level	energy of single-particle levels	final energy of level
1	0.00	—0.520	0.00	—0.500	0.150	—0.280	0.570	0,210
2	1.070	0,460	0.150	—0,340	0.570	0.180	0,720	0,510
3	1.720	1.040	0,920	0.200	0.720	0,370	1.070	0,590

Table II. Energy levels of Tl^{206} and Bi^{210} for different values of the parameters in the interaction potential*

Tl^{206}					Bi^{210}				
Number of level	$I = 0^-$	$I = 1^-$	$I = 2^-$	$I = 3^-$	Number of level	$I = 0^-$	$I = 1^-$	$I = 2^-$	$I = 9^-$
$v_t = 20,\ v_s = 20,\ \hbar\omega = 3,\ C = 1000$					$v_t = 20,\ v_s = 20,\ \hbar\omega = 3,\ C = 1000$				
1	—0,770	—0.810	—0,462	0.010	1	—0,970	—0,690	—0,590	—0,620
2	0.350	—0.490	—0,060	0.300	2	2,600	—0.790	—0,540	—0,270
3	0.980	0.100	0.190	0.510	3	2.890	0.310	0.520	0.570
$v_t = 25,\ v_s = 25,\ \hbar\omega = 3,\ C = 1000$					$v_t = 20,\ v_s = 35,\ \hbar\omega = 3,\ C = 1000$				
1	—0,900	—0,950	—0,580	—0,100	1	—0,970	—1.010	—0.750	—0,760
2	0,190	—0,630	—0.040	0,340	2	2.400	—0,810	—0,560	—0,290
3	0,820	0.240	0.050	0.400	3	2,900	0,050	0.400	0.450
$v_t = 25,\ v_s = 35,\ \hbar\omega = 3,\ C = 1000$					$v_t = 20,\ v_s = 20,\ \hbar\omega = 3,\ C = 2000$				
1	—0.900	—0,880	—0.570	—0.010	1	—0.840	—0.560	—0,500	—0.510
2	0.190	—0.490	—0.025	0,370	2	2,600	—0.670	—0.460	—0.200
3	0.820	0.390	0.060	0,430	3	2.760	0,320	0,520	0.580
$v_t = 20,\ v_s = 35,\ \hbar\omega = 3,\ C = 1000$					$v_t = 20,\ v_s = 35,\ \hbar\omega = 3,\ C = 2000$				
1	—0.770	—0.970	—0.490	—0.140	1	—0.840	—0,910	—0.680	—0.650
2	0.350	—0,600	—0.031	0.330	2	2.400	—0.680	—0.470	—0,230
3	0.980	—0.110	0,170	0.370	3	2,760	0,060	0.410	0.450
$v_t = 20,\ v_s = 20,\ \hbar\omega = 3,\ C = 2000$									
1	—0,650	—0.660	—0,380	0.110					
2	0.310	—0.410	0.120	0.450					
3	1,000	0,120	0.280	0.550					

*The energy is measured from the ground single particle 0^- level (all values are in Mev).

Table III. Expansion coefficients of eigenfunctions of Tl^{206} levels

I	$c^I\ (l_1j_1,\ l_2j_2.\ J;\ NR)$					
	$p_{1/2}, s_{1/2}, 0; 00$	$p_{3/2}, d_{3/2}, 0; 00$	$f_{5/2}; d_{5/2}, 0; 00$	$p_{1/2}, d_{3/2}, 2; 12$	$f_{5/2}, s_{1/2}, 2; 12$	
0_1	0.953	—0.102	0.134	0.159	0,194	
	$p_{1/2}, s_{1/2}, 1; 00$	$p_{1/2}, d_{3/2}\ 1; 00$	$p_{3/2}, s_{1/2}, 1; 00$	$f_{5/2}, d_{3/2}, 1; 00$	$p_{3/2}, d_{3/2}, 1; 00$	$f_{5/2}, d_{5/2}\ 1,; 00$
1_1	0.914	0.034	0.008	—0.260	0.093	0.103
1_2	—0.030	0.899	0.247	0.110	0.251	0.019
	$p_{1/2}, s_{1/2}, 1; 12$	$p_{1/2}, d_{3/2}, 1; 12$	$p_{1/2}, d_{3/2}, 2; 12$	$f_{5/2}, s_{1/2}, 2; 12$	$f_{5/2}, s_{1/2}, 3; 12$	
1_1	—0.006	—0.135	0.137	0.086	0.177	
1_2	—0.166	0.114	0.109	0,021	—0.026	
	$p_{1/2}, d_{3/2}, 2; 00$	$f_{5/2}, s_{1/2}, 2; 00$	$f_{5/2}, d_{3/2}, 2; 00$	$p_{3/2}, d_{3/2}. 2; 00$	$f_{5/2}, d_{5/2}, 2; 00$	$p_{1/2}, s_{1/2}, 0; 12$
2_1	0.898	0.279	0.209	—0.101	0.039	0.112
2_2	0.379	0.749	0.504	0,013	0.072	0.061
	$p_{1/2}, s_{1/2}, 1; 12$	$p_{1/2}, d_{3/2}, 1; 12$	$p_{1/2}, d_{3/2}, 2; 12$	$f_{5/2}, s_{1/2}, 2; 12$	$f_{5/2}, s_{1/2}, 3; 12$	
2_1	—0,132	—0.083	0.131	0.047	—0,038	
2_2	—0,033	—0.035	—0,004	0.136	—0.107	
	$f_{5/2}, s_{1/2}, 3; 00$	$f_{5/2}, d_{3/2}, 3; 00$	$p_{3/2}, d_{3/2}, 3; 00$	$p_{1/2}, d_{5/2}, 3; 00$	$f_{5/2}, d_{5/2}, 3; 00$	$p_{1/2}, s_{1/2}, 1; 12$
3_1	0.855	—0.317	0.031	0.267	0.166	0.199
3_2	0.314	0.877	0.289	0	0.015	0.068
	$p_{1/2}, d_{3/2}, 1; 12$	$p_{1/2}, d_{3/2}, 2; 12$	$f_{5/2}, s_{1/2}, 2; 12$	$f_{5/2}, s_{1/2}, 3; 12$		
3_1	—0,031	0.065	0.038	0.160		
3_2	0,148	—0.091	0.104	—0.055		

lows that the values of the parameters $\hbar\omega$ and C were chosen correctly.

The magnetic transitions $1^- \rightarrow 0^-$ and $1^- \rightarrow 1^-$ are forbidden on the single particle model ($d_{3/2} \rightarrow s_{1/2}$), and the reduced probabilities for M1 transitions differ from zero only because of the admixture of other states. But nevertheless the theoretical value of B(M1) is found to be 10^3 times as great as the experimental value. Variation of the parameters v_t and v_s within reasonable limits enables us to reduce the value of B(M1), but by no more than an order of magnitude. With the same values of v_t and v_s with which the theoretical values of B(M1) agree with experiment, one cannot obtain either the observed spectrum or the binding energy of a pair.

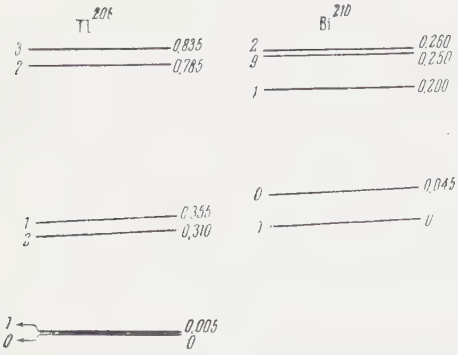

FIG. 1. Level schemes of Tl^{206} and Bi^{210} for the following parameter values: Tl^{206} for $v_t = 22$, $v_s = 27$, $\hbar\omega = 3$, $C = 1000$; Bi^{210} for $v_t = 20$, $v_s = 35$, $\hbar\omega = 3$, $C = 1000$ (all values in Mev).

Consequently, on this point there is disagreement with the results of reference 3.

3. SPECTRUM OF Bi^{210}

The nucleon configurations in Bi^{210} are a combination of the single particle levels of Pb^{209} and Bi^{209}. The multiplets found are characterized by a high degree of degeneracy, which is lifted when the interactions V_S and V_P are included. When this is done, there is a splitting and mixing of levels. Our problem consists in determining parameters which give agreement with the experimental spectrum.

First one must make sure that the 1^- level lies lower than the 0^- level. Under the influence of both V_S and V_P, the 0^- level is pushed farthest down of all the levels in the multiplet $(g_{9/2}, h_{9/2})$. This is a general result, valid for any values of the parameters of the potential. From this it follows that the 1^- level belongs to some other multiplet, namely the multiplet $(i_{11/2}, h_{9/2})$. But in order to become the ground state, the 1^- level must be pushed down 800 kev more than the 0^- level, which is impossible for any reasonable values of the parameters v_t and v_S.

The 1^- level may be lowest, if the single particle level $i_{11/2}$ is lower in Bi^{210} than in Pb^{209}. Here we should recall one property of the $i_{11/2}$ level.[2] It appears that the $i_{11/2}$ level is very sensitive to changes in the diffuseness parameter α in the potential V_C. Increasing the parameter α by 0.1 pushes the $i_{11/2}$ level down by 0.5 Mev, whereas the other levels are practically not shifted. Therefore one can assume that when a proton is added to Pb^{209} to form the Bi^{210} nucleus, there is a slight change in the diffuseness of the boundary of the potential for the neutron, as a result of which the $i_{11/2}$ level approaches the $g_{9/2}$ level. In these computations the $i_{11/2}$ single particle level of Pb^{209} was pushed down by 0.500 Mev.

Table IV. Expansion coefficients of eigenfunctions of Bi^{210} levels

I	$c^I\,(l_1j_1,\ l_2j_2,\ J;\ NR)$				
	$g_{9/2}, h_{9/2}, 0; 00$	$g_{9/2}, h_{9/2}, 2; 12$	$i_{11/2}, h_{9/2}, 2; 12$		
0_1	0,966	—0,259	0.032		
	$g_{9/2}, h_{9/2}, 1; 00$	$i_{11/2}, h_{9/2}, 1; 00$	$g_{9/2}, f_{7/2}, 1; 00$	$g_{9/2}, h_{9/2}, 1; 12$	$g_{9/2}, h_{9/2}, 2; 12$
1_1	—0.237	0,934	0.019	—0,062	0.020
1_2	0.934	0.238	0.043	0,166	0,006
	$g_{9/2}, h_{9/2}, 3; 12$	$i_{11/2}, h_{9/2}, 1; 12$	$i_{11/2}, h_{9/2}, 2; 12$	$i_{11/2}, h_{9/2}, 3; 12$	
1_1	0.038	—0.083	0.178	—0,163	
1_2	—0,193	—0.044	0.053	—0.017	
	$g_{9/2}, h_{9/2}, 2; 00$	$i_{11/2}, h_{9/2}, 2; 00$	$g_{9/2}, f_{7/2}, 2; 00$	$g_{9/2}, h_{9/2}, 2; 12$	$g_{9/2}, h_{9/2}, 3; 12$
2_1	—0.490	0.840	—0,017	—0.082	0,016
	$g_{9/2}, h_{9/2}, 4; 12$	$i_{11/2}, h_{9/2}, 2; 12$	$i_{11/2}, h_{9/2}, 3; 12$	$i_{11/2}, h_{9/2}, 4; 12$	
2_1	0.083	0,070	0.094	—0.160	
	$g_{9/2}, h_{9/2}, 9; 00$	$i_{11/2}, h_{9/2}, 9; 00$	$i_{11/2}, f_{7/2}, 9; 00$	$g_{9/2}, h_{9/2}, 7; 12$	$g_{9/2}, h_{9/2}, 8; 12$
9_1	0.967	0	0.006	—0.042	0
	$g_{9/2}, h_{9/2}, 9; 12$	$i_{11/2}, h_{9/2}, 7; 12$	$i_{11/2}, h_{9/2}, 8; 12$	$i_{11/2}, h_{9/2}, 9; 12$	$i_{11/2}, h_{9/2}, 10; 12$
9_1	0,249	—0,001	—0.005	—0.014	—0,031

Table V. Reduced probabilities of transitions for levels of Tl^{206}

Type of transition	Transition	B			
		single particle	collective	total	experiment
$E2$	$2_1 \rightarrow 0_1$	$0,266 \cdot 10^{11}$	$0,659 \cdot 10^{11}$	$1,76 \cdot 10^{11}$	$1,6 \cdot 10^{11}$
$E2$	$2_1 \rightarrow 1_1$	$0,196 \cdot 10^{11}$	$0,981 \cdot 10^{11}$	$2,05 \cdot 10^{11}$	$2,3 \cdot 10^{11}$
$M1$	$1_2 \rightarrow 1_1$	$2,13 \cdot 10^{12}$	—	$2,13 \cdot 10^{12}$	$0.17 \cdot 11^{10}$
$M1$	$1_2 \rightarrow 0_1$	$1.89 \cdot 10^{12}$	—	$1,89 \cdot 10^{12}$	$0.33 \cdot 10^{10}$

Of the levels with larger values of the spin I, the one which is pushed farthest down is the $I = 9^-$ level; this is also a general result, independent of choice of parameters.

In Fig. 1 is shown the theoretical level scheme of Bi^{210}, calculated for the parameter values $v_t = 20$ Mev and $v_s = 35$ Mev which correspond to the experimentally measured energy difference between the levels $I = 1_1^-$ and $I = 0_1^-$, and also to the binding energy of the neutron-proton pair in the ground state as found from the $Pb^{210} \rightarrow Bi^{210}$ β decay. The existence of a second $I = 1_2^-$ level near the $I = 9^-$ level is not in contradiction with experiment, since this level would be difficult to detect because of the large spin difference and the low energy of the γ transition from the $I = 9^-$ level to the $I = 1_2^-$ level.

Using the wave functions for the various states of Bi^{210} which were found from diagonalization of the energy matrix, we calculated the probability of M1 transition from the $I = 0^-$ level to the level with $I = 1_1^-$. In Table VI we give the reduced and total probabilities of M1 transition as a function of the character of the $I = 1_1^-$ level. The mixture of the multiplets $(i_{11/2}, h_{9/2})$ and $(g_{9/2}, h_{9/2})$ used here corresponds to the optimum parameter values. From Table VI we see that the probability of this transition is very sensitive to the composition of the mixture. An exact experimental determination of the probability of the M1 transition would make it possible to determine the character of the $I = 1_1^-$ level. It should be mentioned that if the parameter C is doubled (so that C = 2000 Mev) the energy spectrum is not changed, but the binding energy of the pair is reduced by 150 kev.

Table VI. Probability of transition between levels $0_1^- \rightarrow 1_1^-$ for different assumptions concerning the composition of the 1_1^- level

Character of 1_1^- level	B, sec^{-1}	T, sec^{-1}
Pure $i_{11/2}h_{9/2}$ state	0	0
Pure $g_{9/2}h_{9/2}$ state	$1 \cdot 10^{14}$	$9.1 \cdot 10^{9}$
Mixed state, found in this work	$6.4 \cdot 10^{12}$	$5.8 \cdot 10^{8}$

4. GENERAL TREATMENT OF THE n-p INTERACTION

From the calculations presented above it follows that the effects produced by the pairing interaction in the neutron-proton system differ essentially from the corresponding effects in a system of two identical particles. Whereas for a system of identical particles the main contribution to the pairing interaction comes from singlet forces, in the n-p system the triplet forces play the main role. Whereas in the n-n (or p-p) system the ground state (0^+) is separated from the other states by a sizable energy gap, in the n-p system the pairing energy for different values of I varies more smoothly and according to a different law. In addition, since there may be one or two levels with different spins between the $I = 0^-$ and $I = 2^-$ levels (for example, for Bi^{210}, $I = 1^-$ and $I = 9^-$), the average distance between levels is 100 — 150 kev.

For a qualitative treatment of the role of singlet and triplet interactions in the n-p system, it is convenient to use δ function interactions. The operator for the pairing interaction can be written as

$$V_{12} = V_{12}^t + V_{12}^s = -\delta(\mathbf{r}_1 - \mathbf{r}_2)[v_t\pi_t + v_s\pi_s]. \tag{7}$$

The general nature of the splitting of levels of a given multiplet can be understood, starting from the treatment of the diagonal matrix elements of the n-p interaction, which are easily calculated using the algebra of irreducible tensor operators. For states of negative parity, which are most usual for heavy nuclei, the matrix elements of the triplet interaction V_{12}^t and the singlet interaction V_{12}^S are equal to:

a) for even values of the angular momentum J

$$\langle|V_{12}^t|\rangle = -\tfrac{1}{2}v_tF\langle l_1j_1\|T^J\|l_2j_2\rangle^2\{1 + [4J(J+1)]^{-1} \times [(2j_1+1) + (-1)^{j_1+j_2}(2j_2+1)]^2\}, \tag{8a}$$

$$\langle|V_{12}^s|\rangle = 0; \tag{9a}$$

b) for odd values of the angular momentum J

$$\langle|V_{12}^t|\rangle = -\tfrac{1}{2}v_tF\langle l_1j_1\|T^J\|l_2j_2\rangle^2[4J(J+1)]^{-1} \times [(2j_1+1) - (-1)^{j_1+j_2}(2j_2+1)]^2, \tag{8b}$$

$$\langle|V_{12}^s|\rangle = -\tfrac{1}{2}v_sF\langle l_1j_1\|T^J\|l_2j_2\rangle^2. \tag{9b}$$

Here F is the Slater integral, and

$$\langle l_1j_1\|T^J\|l_2j_2\rangle^2 = \tfrac{1}{2}[1 + (-1)^{l_1+l_2+J}] \times [(2j_1+1)(2j_2+1)/(2J+1)][C_{j_1 1/2\, j_2 -1/2}^{J0}]^2, \tag{10}$$

where the coefficients $C_{j_1 \frac{1}{2} j_2 -\frac{1}{2}}^{J_0}$ for $j = 1/2, 3/2, \ldots 9/2$ are tabulated in the paper of de Shalit.[4]

Computations made for the case of $v_s = 2v_t$ show that in both case a) and case b) the dependence of the pairing energy can be described by the two curves shown in Fig. 2. For a multiplet of the type $j_1 = l_1 \pm 1/2$, $j_2 = l_2 \mp 1/2$, curve a describes the value of the pairing energy for even values of the angular momentum J, while curve b applies

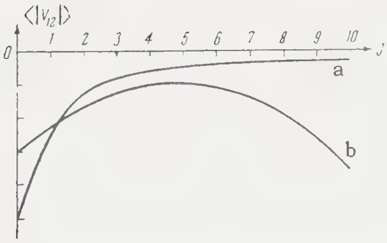

FIG. 2. Dependence of np-interaction energy on angular momentum J.

for odd J. For multiplets of the type $j_1 = l_1 \pm 1/2$, $j_2 = l_2 \pm 1/2$, the reverse holds: curve a is for odd J and curve b for even J. From these curves it follows that, just as in the case of identical particles, the level which is pushed deepest is the one with J_{min}, but the second level can be any other, including the level with J_{max}.

An analysis of results obtained with forces of finite range shows that the general character of the multiplet splitting remains the same as for δ forces. But one finds that the coefficient of v_t is larger, and the coefficient of v_s smaller, than the corresponding coefficients for δ forces. Consequently, for forces of finite range the triplet interaction predominates, even when the main contribution in the case of δ forces comes from the singlet interaction.

In conclusion we should mention that the values of the parameters v_t and v_s for Tl^{206} and Bi^{210} are quite close to one another, even though we are dealing in the first case with an interaction of holes and in the second with an interaction of particles. Additional data on these nuclei will make it possible to fix more precisely the values of the interaction parameters. But it is clear even now that including the n-p pairing interaction gives a complete picture of the properties of the low-lying levels of odd-odd nuclei.

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THEORY OF THE INTERACTION OF A CHARGED PARTICLE WITH A PLASMA IN A MAGNETIC FIELD

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The interaction of a nonrelativistic charged particle with an electron plasma in a magnetic field is studied by quantum field theoretic methods. The dielectric constant, frequencies, and damping coefficients for the longitudinal oscillations of the plasma in the magnetic field are calculated to first order in e^2. A general formula is obtained for the energy loss of a particle moving through a plasma. The case in which the particle moves with a velocity much greater than the mean thermal velocity of the electrons in the plasma is investigated in detail.

1. It is the purpose of the present paper to study the interaction of a nonrelativistic charged particle with an electron plasma in a constant uniform magnetic field.

Because the Coulomb forces are long-range forces, the energy losses of a charged particle which moves through a plasma are determined by remote collisions as well as short-range collisions. The contribution of the short-range collisions is usually found by the pair-collision method; the effect of long-range collisions is generally investigated through the use of the kinetic equation in conjunction with a self-consistent field. In some cases the plasma is regarded as an optically active medium and is characterized by a dielectric constant. The energy losses of a particle moving through a plasma in a magnetic field have been computed by these methods by A. Akhiezer and Faĭnberg (cf. reference 1), Kolomenskii,[2] and Sitenko and Stepanov.[3] However, these methods do not give an accurate description of the interaction of particles which are separated by distances of the order of the Debye radius, so that the results are only of logarithmic accuracy.

In the present paper the energy losses of a particle which passes through a plasma in a magnetic field are computed by quantum field-theoretical methods. In this way we are able to obtain a general formula for the energy losses and, in limiting cases, to find the factors which appear in the argument of the logarithm.*

2. The Hamiltonian of a system of charged particles which interact through Coulomb interaction with an external particle which moves through the system can be written in the form $\mathcal{H} + \mathcal{H}'(t)$, where $\mathcal{H}$ is the basic Hamiltonian,

$$\mathcal{H}'(t) = \int J_0(\mathbf{r}, t)\, a_0(\mathbf{r}, t)\, d\mathbf{r},$$

$$a_0(\mathbf{r}, t) = (4\pi)^{-1} \int |\mathbf{r} - \mathbf{r}'|^{-1}\, j_0(\mathbf{r}', t)\, d\mathbf{r}', \qquad (1)$$

$\mathcal{H}'$ is the interaction Hamiltonian, a_0 is the scalar-potential operator and j_0 and J_0 are operators describing the charge density of the system and of the incident particle; the operators are taken in the interaction representation.

Using $\mathcal{H}'(t)$, we form the scattering matrix

$$S = T \exp\left\{-i \int_{-\infty}^{\infty} \mathcal{H}'(t)\, dt\right\} \ldots$$

and determine the matrix elements, which couple the different states of the initial system, i.e., the medium plus the external particle. We characterize these states by the quantum numbers α and n, where $\alpha \equiv (\nu, p_z, q)$ represents the quantum numbers characterizing the incident particle in the magnetic field H, while n represents the ensemble of quantum numbers which describe the state of the medium with a given energy E_n and a definite number of particles N_n.

We shall assume that the velocity of the incident particle V is large $(e^2 V^{-1} \ll 1)$ so that its interaction with particles in the medium can be analyzed by perturbation theory. In the linear approximation in $\mathcal{H}'$, the probability for transition of the system from a state α, n to a state α', n′ is given by

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*Quantum field theoretical methods have been used to study the interaction of a particle with a plasma (no magnetic field) in a paper by Larkin.[4]

$$2\pi\delta\,(E_n - E_{n'} + \mathscr{E}_\alpha - \mathscr{E}_{\alpha'}) \int d\mathbf{r}\, d\mathbf{r}' \langle \alpha' | \hat{J}_0(\mathbf{r}) | \alpha \rangle \langle \alpha | \hat{J}_0(\mathbf{r}') | \alpha' \rangle$$
$$\times \langle n' | \hat{a}_0(\mathbf{r}) | n \rangle \langle n | \hat{a}_0(\mathbf{r}') | n' \rangle, \qquad (2)$$

where $\mathscr{E}_\alpha = \mathscr{E}_{\nu,p_z} = \eta(\nu + 1/2)m/M + p_z^2/2M$ is the energy of the incident particle, M is its mass and $\eta = eH/mc$ is the Larmor frequency of the electron in the magnetic field H ($\hat{J}_0$ and $\hat{a}_0$ are operators in the Schrödinger representation).

In order to obtain the total probability for transition of the particle from a state characterized by energy $\mathscr{E}_{\nu p_z}$ to a state with $\mathscr{E}_{\nu' p_z'}$, we sum Eq. (2) over final states of the medium and average over initial states using the density matrix $\rho_0 = \exp\{\beta(\Omega + \mu N - \mathscr{H})\}$ (β is the reciprocal temperature, Ω is the thermodynamic potential, N is the operator describing the number of particles, μ is the chemical potential); we also sum over all values of the variable q′ and average over all values of the variable q. In terms of Fourier components we have

$$W_{\nu p_z,\, \nu' p'_z} = 2\pi \int d\mathbf{k}\, \Phi\,(\mathbf{k}, \mathscr{E}_{\nu p_z} - \mathscr{E}_{\nu' p'_z})\, U_{\nu p_z,\, \nu' p'_z}(\mathbf{k}), \qquad (3)$$

where $\Phi(\mathbf{k}, \omega)$ and $U_{\nu p_z, \nu' p'}(\mathbf{k})$ are the components of the Fourier functions

$$\Phi\,(\mathbf{r}_1 - \mathbf{r}_2;\, \omega) = \sum_{n,\, n'} \exp\{\beta\,(\Omega + \mu N_n - E_n)\} \langle n' | \hat{a}_0(\mathbf{r}_1) | n \rangle$$
$$\times \langle n | \hat{a}_0(\mathbf{r}_2) | n' \rangle\, \delta\,(E_n - E_{n'} + \omega), \qquad (4)$$

$$U_{\nu p_z,\, \nu' p'_z}(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{q,\, q'} \langle \alpha' | \hat{J}_0(\mathbf{r}_1) | \alpha \rangle \langle \alpha | \hat{J}_0(\mathbf{r}_2) | \alpha' \rangle. \qquad (5)$$

The energy lost by the particle per unit time is related to W by the expression

$$-\frac{d}{dt} E_{\nu p_z} = \sum_{\nu',\, p'_z} (\mathscr{E}_{\nu p_z} - \mathscr{E}_{\nu' p'_z})\, W_{\nu p_z,\, \nu' p'_z}. \qquad (6)$$

3. We now calculate the function $U_{\nu p_z, \nu' p'_z}(\mathbf{k})$. In this calculation we make use of the expression for the matrix element of the operator $\hat{J}_0(\mathbf{r})$

$$\langle \alpha' | \hat{J}_0(\mathbf{r}) | \alpha \rangle = e \Psi^*_{\alpha'}(\mathbf{r})\, \Psi_\alpha(\mathbf{r}),$$

where the $\Psi_\alpha(\mathbf{r})$ are the wave functions for the particle in the magnetic field

$$\Psi_\alpha(\mathbf{r}) = (2\pi)^{-1} \exp\,(ip_z z + iqy)\, \varphi_\nu(x - q/m\eta) \qquad (7)$$

($\varphi_\nu(x)$ is the wave function describing an oscillator of frequency η).

Noting that

$$\int_{-\infty}^{\infty} d\xi\, \exp\,(i\xi m\eta y)\, \varphi_\nu\left(\xi + \frac{x}{2}\right) \varphi_\nu\left(\xi - \frac{x}{2}\right)$$
$$= \exp\left\{-m\eta \frac{x^2 + y^2}{4}\right\} L_\nu\left(m\eta \frac{x^2 + y^2}{2}\right), \qquad (8)$$

($L_\nu(x)$ is the Laguerre polynomial), we write Eq. (5) in the form

$$U_{\nu p_z,\, \nu' p'_z}(\mathbf{r}) = e^2 \frac{(m\eta)^2}{(2\pi)^4} \exp\left\{i\,(p_z - p'_z)\, z\right\}$$
$$\times \exp\left\{-m\eta \frac{x^2 + y^2}{2}\right\} L_\nu\left(m\eta \frac{x^2 + y^2}{2}\right) L_{\nu'}\left(m\eta \frac{x^2 + y^2}{2}\right).$$

Taking the Fourier transform we have

$$U_{\nu p_z,\, \nu' p'_z}(\mathbf{k}) = e^2 m\eta\, (2\pi)^{-2}\, \delta\,(p_z - p'_z - k_z)\, \Lambda_{\nu\nu'}(k_t/\sqrt{2m\eta}), \qquad (9)$$

where $k_t = \sqrt{k_x^2 + k_y^2}$ and

$$\Lambda_{\nu\nu'}(a) = \int_0^\infty ds\, J_0(2a\sqrt{s})\, L_\nu(s)\, L_{\nu'}(s)\, e^{-s}. \qquad (10)$$

In order to compute the function $\Phi(\mathbf{k}, \omega)$ we relate it to the retarded Green's function for a scalar photon in the medium

$$D\,(\mathbf{r}_1 - \mathbf{r}_2,\, t_1 - t_2)$$
$$= -i\theta\,(t_1 - t_2)\, \mathrm{Sp}\,\{\rho_0\, [a_0(\mathbf{r}_1, t_1),\, a_0(\mathbf{r}_2, t_2)]\},$$
$$\theta(t) = 0;\ 1 \text{ for } t \gtrless 0. \qquad (11)$$

For this purpose we use the spectral representation of the components of the Fourier function D

$$D\,(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \frac{\rho\,(\mathbf{k}, \omega')}{\omega' + \omega + i0}\, d\omega',$$

where

$$\rho\,(\mathbf{k}, \omega) = \sum_{n,\, n'} [\exp\{\beta\,(\Omega + \mu N_n - E_n)\}$$
$$- \exp\{\beta\,(\Omega + \mu N_{n'} - E_{n'})\}]$$
$$\times \langle n | \hat{a}_0(\mathbf{k}) | n' \rangle \langle n' | \hat{a}_0(\mathbf{k}) | n \rangle\, \delta$$
$$\times (E_n - E_{n'} - \mu N_n + \mu N_{n'} + \omega)$$
$$= (e^{\beta\omega} - 1) \sum_{n,\, n'} \exp\,\{\beta\,(\Omega + \mu N_{n'} - E_{n'}\} \langle n | \hat{a}_0(\mathbf{k}) | n' \rangle$$
$$\times \langle n' | \hat{a}_0(\mathbf{k}) | n \rangle\, \delta\,(E_n - E_{n'} + \omega)$$

(we may note that the only nonvanishing matrix elements for the potential a_0 are those between states with the same number of particles: $N_n = N_{n'}$). Comparing this expression with Eq. (4) we obtain the following relation for Φ

$$\Phi\,(\mathbf{k}, \omega) = (e^{-\beta\omega} - 1)^{-1}\, \rho\,(\mathbf{k}, -\omega)$$
$$= \frac{2}{2\pi} (1 - e^{-\beta\omega})^{-1}\, \mathrm{Im}\, D\,(\mathbf{k}, \omega). \qquad (12)$$

The function $D(\mathbf{k}, \omega)$ is the analytic continuation of the temperature Green's function $\mathscr{D}(\mathbf{k}, k_4)$:

$$\mathscr{D}(\mathbf{k}, k_4) = \frac{1}{2} \int_{-\beta}^{\beta} d\tau\, \mathrm{Sp}\,\{\rho_0 T_\tau\, [a_0(\mathbf{k}, \tau)\, a_0(\mathbf{k}, 0)]\}\, e^{ik_4\tau} \qquad (13)$$

$[a_0(\mathbf{k}, \tau) = e^{\tau(\mathscr{H} - \mu N)}\, \hat{a}_0(\mathbf{k})\, e^{-\tau(\mathscr{H} - \mu N)}]$ and is related to the temperature polarization operator $\mathscr{P}$ by the expression

$$\mathscr{D}(\mathbf{k}, k_4) = -\{k^2 - \mathscr{P}(\mathbf{k}, k_4)\}^{-1}. \qquad (14)$$

Introducing the dielectric constant $\epsilon(\mathbf{k}, \omega) = 1 + \kappa(\mathbf{k}, \omega)$, where

$$\varkappa(\mathbf{k}, \omega) = -k^{-2}\mathscr{P}(\mathbf{k}, i\omega - 0), \qquad (15)$$

we write Eq. (12) in the form

$$\Phi(\mathbf{k}, \omega) = \frac{1}{\pi} k^{-2}(1 - e^{-\beta\omega})^{-1} \operatorname{Im} \frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)}. \qquad (16)$$

Substituting Eqs. (9) and (16) in Eq. (6), we obtain the following general formula for the energy loss of a charged particle which moves through a system of charged particles in a magnetic field:

$$-\frac{d}{dt} E_{\nu p_z} = \frac{2e^2 m\eta}{(2\pi)^2} \sum_{\nu'} \int_{-\infty}^{\infty} \frac{\omega d\omega}{1 - e^{-\beta\omega}} \int \frac{d\mathbf{k}}{k^2} \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)$$
$$\times \operatorname{Im} \frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)} \delta\left(\mathscr{E}_{\nu' p_z} - \mathscr{E}_{\nu', p_z - k_z} - \omega\right). \qquad (17)$$

4. We now compute the electric susceptibility κ of a plasma in a magnetic field. Equation (15) relates κ to the reduced polarization operator $\mathscr{P}(\mathbf{k}, k_4)$; this operator is calculated by means of graphs. In the first approximation in e^2

$$\mathscr{P}(\mathbf{r}_1 - \mathbf{r}_2; k_4) = \frac{2e^2}{\beta} \sum_{p_4} \mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4)\, \mathscr{G}(\mathbf{r}_2, \mathbf{r}_1; p_4 - k_4),$$
$$p_4 = \frac{2n+1}{\beta}\pi, \quad n = 0, \pm 1, \qquad (18)$$

where the Green's function for the electron in the magnetic field $\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2, p_4)$ is of the form

$$\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4) = \sum_\alpha \Psi_\alpha(\mathbf{r}_1) \frac{1}{\varepsilon_\alpha - \mu + ip_4} \Psi^*_\alpha(\mathbf{r}_2), \qquad (19)$$

$\epsilon_\alpha \equiv \epsilon_{\nu, p_z} = \eta(\nu + 1/2) + p_z^2/2m$ and the function $\Psi_\alpha(\mathbf{r})$ is given by (7) (we neglect the interaction between the electron spin and the magnetic field).

Using Eq. (8), we write Green's function for the electron in the form

$$\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4) = \frac{m\eta}{(2\pi)^2} \exp\left\{im\eta \frac{x_1 + x_2}{2}(y_1 - y_2)\right\}$$
$$\times \sum_{\nu=0}^{\infty} \int_{-\infty}^{\infty} dp_z \exp\{ip_z(z_1 - z_2)\}$$
$$\times \exp\left\{-m\eta \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4}\right\}$$
$$\times L_\nu\left\{m\eta \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2}\right\} \frac{1}{\varepsilon_{\nu p_z} - \mu + ip_4}. \qquad (20)$$

Substituting (20) in (18) and taking Fourier components we have

$$\mathscr{P}(\mathbf{k}, k_4) = \frac{2e^2 m\eta}{(2\pi)^2} \sum_{\nu, \nu'} \int_{-\infty}^{\infty} dp_z \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)$$
$$\times \frac{1}{\beta} \sum_{p_4} \frac{1}{\varepsilon_{\nu p_z} - \mu + ip_4} \frac{1}{\varepsilon_{\nu', p_z - k_z} - \mu + i(p_4 - k_4)}.$$

The summation over p_4 is carried out using the relation

$$\frac{1}{\beta} \sum_{p_4} (\varepsilon_\alpha - \mu + ip_4)^{-1} = \frac{1}{2} - n_\alpha,$$

where $n_\alpha = n_{\nu p_z} = [\exp\{\beta(\epsilon_{\nu p_z} - \mu)\} + 1]^{-1}$ is the Fermi distribution function. Taking account of the relation between κ and $\mathscr{P}$ (15), we obtain the final expression for the electric susceptibility of the electron gas in the magnetic field:

$$\varkappa(\mathbf{k}, \omega) = -\frac{2e^2 m\eta}{(2\pi)^2} \frac{1}{k^2}$$
$$\times \sum_{\nu, \nu'} \int_{-\infty}^{\infty} dp_z \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right) \frac{n_{\nu p_z} - n_{\nu', p_z - k_z}}{\varepsilon_{\nu p_z} - \varepsilon_{\nu', p_z - k_z} - \omega - i0}. \qquad (21)$$

In a weak magnetic field ($k_t^2/2m\eta \gg 1$ or $k_t\bar{p}/2m\eta \gg 1$; $\bar{p}$ is the mean momentum of the plasma electrons) the summation over ν and ν' is carried out by means of Eq. (A.2) of the Appendix. We obtain the following familiar expression for $\kappa(\mathbf{k}, \omega)$

$$\varkappa(\mathbf{k}, \omega) = -\frac{2e^2}{(2\pi)^3} \frac{1}{k^2} \int d\mathbf{p} \frac{n_\mathbf{p} - n_{\mathbf{p}-\mathbf{k}}}{\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega - i0}, \qquad (22)$$

where $\epsilon_\mathbf{p} = p^2/2m$ and $n_\mathbf{p} = [\exp\{\beta(\epsilon_\mathbf{p} - \mu)\} + 1]^{-1}$.

In a strong magnetic field ($k_t^2/2m\eta \ll 1$ and $k_t\bar{p}/2m\eta \ll 1$) we use Eq. (A.2) of the Appendix to compute $\kappa(\mathbf{k}, \omega)$. In this case we have when $k \ll \bar{p}$

$$\operatorname{Re}\varkappa(\mathbf{k}, \omega) = -\frac{\Omega^2}{\omega^2}\cos^2\theta - \frac{\Omega^2}{\omega^2 - \eta^2}\sin^2\theta,$$
$$\operatorname{Im}\varkappa(\mathbf{k}, \omega) = \Omega^2 \frac{(2\pi m\beta)^{3/2}\omega}{4\pi k^3\cos\theta}\left\{\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega}{k\cos\theta}\right)^2\right\}\left(1 - \frac{k^2\sin^2\theta}{m\beta\eta^2}\right)\right.$$
$$+ \frac{k^2\sin^2\theta}{2m\beta\eta^2}\left[\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega - \eta}{k\cos\theta}\right)^2\right\}\right.$$
$$\left.\left. + \exp\left\{-\frac{\beta m}{2}\left(\frac{\omega + \eta}{k\cos\theta}\right)^2\right\}\right]\right\}, \qquad (23)$$

where Ω is the electron Langmuir frequency

$$\Omega^2 = \frac{e^2 N}{m}, \quad N = \frac{2m\eta}{(2\pi)^2} \sum_\nu \int_{-\infty}^{\infty} dp_z n_{\nu p_z}$$

and θ is the angle formed by the vector $\mathbf{k}$ and the magnetic field [the second equation in (23) is obtained under the assumption that the electron gas is nondegenerate: $\beta\mu \ll 1$; the first expression is obtained under the assumption that $\omega/k \gg \bar{p}/m$].

The poles of the function $D(\mathbf{k}, \omega)$ determine the dispersion relation for the characteristic longitudinal oscillations of the plasma. From (14) and (15) we have $D(\mathbf{k}, \omega) = -\{k^2\epsilon(\mathbf{k}, \omega)\}^{-1}$; thus we obtain the dispersion equation

$$\varepsilon(\mathbf{k}, \omega) \equiv 1 + \varkappa(\mathbf{k}, \omega) = 0. \qquad (24)$$

The oscillation frequencies and damping factors $\omega = \omega_{1,2} + i\gamma_{1,2}$ are then

$$\omega_{1,2}^2 = \frac{1}{2}(\Omega^2 + \eta^2) \pm \frac{1}{2}\sqrt{(\Omega^2 + \eta^2)^2 - 4\Omega^2\eta^2\cos^2\theta},$$

$$\gamma_{1,2} = \Omega^2 \frac{(2\pi m\beta)^{3/2}\omega_{1,2}^2}{8\pi k^3 \cos\theta}\left[1 + \frac{\Omega^2\eta^2\sin^2\theta}{(\omega_{1,2}^2 - \eta^2)^2}\right]^{-1}$$

$$\times\left\{\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2}}{k\cos\theta}\right)^2\right\} + \frac{k^2\sin^2\theta}{2m\beta\eta^2}\right.$$

$$\times\left[\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2} - \eta}{k\cos\theta}\right)^2\right\}\right.$$

$$\left.\left. + \exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2} + \eta}{k\cos\theta}\right)^2\right\}\right]\right\}, \tag{25}$$

in agreement with the results obtained in references 5 and 6.

5. We now find the energy losses of a particle which moves through a nondegenerate plasma ($\beta\mu \ll 1$) with a velocity V which is much larger than the mean thermal velocity of the electrons in the plasma. If $\eta/\Omega \ll m^i/m$ (m^i is the ion mass), then the interaction of the particle with the ions in the plasma can be neglected. It will be assumed that the magnetic field satisfies the inequality $\beta\eta \ll 1$.

We divide the integral over **k** in Eq. (17) into two regions: $k < k_1$ and $k > k_1$, where k_1 is chosen so that $k_0 \ll k_1 \ll \overline{p}$ ($k_0 = \sqrt{e^2N\beta}$ is the reciprocal of the Debye radius). In calculating the contribution from short-range collisions ($k > k_1$) the curvature of the Larmor orbit of the incident particle can be described approximately by the momentum **p**. Using Eq. (A.2) we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_c = \frac{2e^2}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}$$

$$\times \int_{k_1}^{\infty} dk \int do_{\mathbf{k}} \operatorname{Im}\frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)}\delta\left(\frac{p^2}{2M} - \frac{(\mathbf{p} - \mathbf{k})^2}{2M} - \omega\right).$$

(We assume that the unit volume in the number space ν, p_z corresponds to the volume $(2\pi m\eta)^{-1}$ in **p**-space).

The electrical susceptibility $\kappa(\mathbf{k}, \omega)$ in this region of wave vectors being considered is given by Eq. (22); noting that κ is of order k_0^2/k^2, we can neglect this quantity compared with unity. Neglecting terms of order $\overline{p}/mV$, we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_c = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV}{k_1(M + m)}. \tag{26}$$

We now compute the contribution due to long-range collisions ($k < k_1$). If one of the following conditions is satisfied:

$$\sin\alpha \ll \frac{m}{M}\frac{\eta}{max\{\Omega, \eta\}} \quad \text{or} \quad \sin\alpha \gg \frac{m}{M}\frac{\eta}{max\{\Omega, \eta\}}$$

(α is the angle between the velocity of the incident particle and the magnetic field), then the state of the incident particle can be described approximately by the momentum **p**. (The first condition corresponds to motion of the particle along the magnetic field, while the second means that the curvature of the Larmor orbit of the particle is small). Using (A.1) and (A.2) we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{2e^2}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}$$

$$\times \int_0^{k_1} dk \int do_{\mathbf{k}} \operatorname{Im}\frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)}\delta\left(\frac{\mathbf{pk}}{M} - \omega\right).$$

Noting that the main contribution in this integral is given by the circuit around the poles of the integrand lying in the lower half plane of the complex variable ω close to the points $\omega_{1,2}$ [cf. Eq. (25)] we have

$$-\left(\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2}{(2\pi)^2 V}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}\int_0^{k_1}\frac{dk}{k}\int do_{\mathbf{k}}\,|\omega^2 - \eta^2|$$

$$\times\,\delta\left(\omega^2 - \eta^2 - \Omega^2 + \frac{\Omega^2\eta^2}{\omega^2}\cos^2\theta\right)$$

$$\times\,\delta\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi - \frac{\omega}{Vk}\right)$$

(θ and φ are the polar angles of the vector **k**). Finally, carrying out the integration over dk and $do_{\mathbf{k}}$, and neglecting terms of order $\overline{p}/mV$, we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2\Omega^2}{4\pi V}\left\{f_1\left(\alpha, \frac{\Omega}{\eta}\right)\ln\frac{k_1V}{\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\},$$

where

$$f_1(\alpha, u) = \frac{1}{\pi u^2}\left\{\int_0^{z_1} g(z)\,dz - \int_{z_2}^{z_3} g(z)\,dz\right\},$$

$$g(z) = \frac{z(1 - z)}{\sqrt{z(z - z_1)(z - z_2)(z_3 - z)}},$$

$$f(\alpha, u) = \frac{1}{\pi u^2}\left\{\int_0^{z_1} g(z)\ln z\,dz - \int_{z_2}^{z_3} g(z)\ln z\,dz\right\},$$

$$z_{1,2} = \frac{1 + u^2}{2} \mp \frac{1}{2}\sqrt{(1 + u^2)^2 - 4u^2\sin^2\alpha}, \qquad z_3 = 1 + u^2. \tag{27}$$

Since $f_1(\alpha, u) \equiv 1$ (cf. Sec. 3 of the Appendix), the energy loss of the particle due to long-range collisions is:

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{k_1V}{\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\}. \tag{28}$$

Adding this expression to Eq. (26) for $(-dE_{\mathbf{p}}/dt)_c$ we have

$$-\frac{dE_{\mathbf{p}}}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{2mMV^2}{(M + m)\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\}. \tag{29}$$

6. If the particle moves along the magnetic field ($\sin\alpha \ll 1$) then $f(\alpha, u) = \ln\sqrt{1+u^2}$ and the total loss is given by

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV^2}{(M+m)\sqrt{\Omega^2+\eta^2}}. \tag{30}$$

This formula has been obtained (to within logarithmic accuracy) by Sitenko and Stepanov.[3*]

For a strong magnetic field ($\eta \gg \Omega$)

$$f(\alpha, u) \Rightarrow \frac{1}{4}\sin^2\alpha\left\{\frac{1}{4} + \ln\frac{u^2\sin^2\alpha}{4}\right\} \quad (u \ll 1),$$

and the energy-loss formula assumes the form

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{2mMV^2}{(M+m)\eta} - \frac{\sin^2\alpha}{4}\ln\frac{\Omega^2\sin^2\alpha}{4\eta^2} - \frac{\sin^2\alpha}{4}\right\}. \tag{31}$$

If a heavy particle ($M \gg m$) moves perpendicularly to the magnetic field ($\alpha = \pi/2$), then

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{4mV^2}{\sqrt{2\Omega\eta}} - \frac{1}{4}\right\}. \tag{32}$$

In the case of a weak magnetic field ($\eta \ll \Omega$) $f = (\alpha, \Omega/\eta) = \ln(\Omega/\eta)$ and

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV^2}{(M+m)\Omega}. \tag{33}$$

This formula agrees with the energy loss formula obtained by Larkin.[4]

In conclusion the author wishes to thank A. I. Akhiezer for a discussion of this work.

APPENDIX

1. We compute the asymptote of the function $\Lambda_{\nu\nu'}(a)$ [cf. Eq. (10)] as $a \to 0$. Expanding the Bessel function $J_0(2a\sqrt{s})$ in a Taylor series we have

$$\Lambda_{\nu\nu'}(a) = \sum_{k=0}^{\infty}(-1)^k a^{2k}(k!)^{-2}\lambda_{\nu\nu'}^{(k)},$$

$$\lambda_{\nu\nu'}^{(k)} = \int_0^\infty ds\, s^k e^{-s} L_\nu(s) L_{\nu'}(s).$$

Taking account of the fact that

$$\int_0^\infty ds\, e^{-s} L_\nu(s) L_{\nu'}(s) = \delta_{\nu\nu'}$$

and using the recurrence formula for the Laguerre polynomials[8]

$$sL_\nu(s) = -(\nu+1)L_{\nu+1}(s) + (2\nu+1)L_\nu(s) - \nu L_{\nu-1}(s),$$

we can find the successive coefficients $\lambda_{\nu\nu'}^{(k)}$ and compute $\Lambda_{\nu\nu'}(a)$ to the required degree of accuracy. Keeping the first two terms we have

$$\Lambda_{\nu\nu'}(a) = \delta_{\nu\nu'} + a^2\{(\nu+1)\delta_{\nu+1,\nu'} - (2\nu+1)\delta_{\nu\nu'} + \nu\delta_{\nu-1,\nu'}\} \tag{A.1}$$

We may note that the expansion of the function $\Lambda_{\nu\nu'}(a)$ as $a \to 0$ actually reduces (for $\nu \neq 0$) to the quantity $a^2\nu$.

2. We now investigate a sum of the form

$$\sum_{\nu'}\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)F(\nu, \nu') \quad \text{for} \quad \frac{k_t^2(2\nu+1)}{2m\eta} \gg 1$$

Using the asymptote for the Laguerre polynomials[8]

$$\exp\left(-\frac{s}{2}\right)L_\nu(s) = J_0\left(2\sqrt{s(\nu+{}^1\!/_2)}\right) + O(\nu^{-3/4}),$$

we have

$$\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right) \to m\eta\int_0^\infty \rho\, d\rho\, J_0(k_t\rho)\, J_0(p_t\rho)\, J_0(p_t'\rho) = \frac{2m\eta}{2\pi}\frac{1}{\Delta},$$

where Δ is the area of the triangle formed by from the segments k_t, p_t and p_t'; if this triangle cannot be formed, then $\Delta^{-1} = 0$ [we use the notation $p_t^2 = 2m\eta(\nu + 1/2)$, $p_t'^2 = 2m\eta(\nu' + 1/2)$]. Noting that the quantity Δ^{-1} can be written in the form

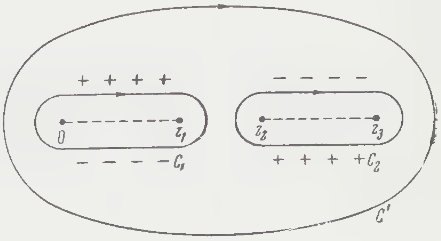
$$\Delta^{-1} = \frac{1}{2}\int_0^{2\pi} d\varphi\, \delta(\mathbf{p}_t' - \mathbf{p}_t + \mathbf{k}_t),$$

where φ is the angle between the plane vectors $\mathbf{p}_t'$ and $\mathbf{k}_t$, we have

$$\sum_{\nu'}\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)F(\nu, \nu') \to \frac{1}{2\pi}\int d\mathbf{p}_t'\,\delta(\mathbf{p}_t' - \mathbf{p}_t + \mathbf{k}_t)\,F(\nu, \nu')\Big|_{\substack{\nu+1/2 = p_t^2/2m\eta,\\ \nu'+1/2 = p_t'^2/2m\eta.}} \tag{A.2}$$

3. We compute the function $f_1(\alpha, u)$ [cf. Eq. (27)]. In the plane of the complex variable z we take a cut along the real axis along the segments $(0, z_1)$ and (z_2, z_3). The function $g(z)$ is a single-valued analytic function in the divided planes; we define this function such that $g(z) > 0$ along the upper edge of the cut $(0, z)$. The quantity $f_1(\alpha, u)$ can be written in the form

$$f_1(\alpha, u) = \frac{1}{2\pi u^2}\int_C [g(z) - i]\, dz,$$



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[*] It has come to the author's attention that Eq. (30) has also been obtained by V. Gurevich and Firsov by quantum field-theoretical methods.[7]

where the contour C consists of the two closed contours C_1 and C_2 (cf. figure) and the integrand vanishes at infinity. Using Cauchy's theorem we replace the integration over the contour C by integration over the infinitely remote contour C′. Since the residue of the function $[g(z) - i]$ is $i[1 - (z_1 + z_2 + z_3)/2]$ at an infinitely remote point, we find $f_1(\alpha, u) = 1$.

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STABILITY CONDITIONS ON THE ELECTRON DISTRIBUTION FUNCTION FOR A PLASMA

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General conditions for the stability of the electron distribution function for a plasma with respect to high-frequency plasma oscillations are deduced neglecting collisions. Free plasma and plasma immersed in a constant, uniform electric or magnetic field are considered.

1. It is known that the mean free path of particles in a plasma and the equilibration time increase with rising temperature. Consequently, in investigating a high-temperature plasma (as well as a very rarefied one) one has to deal with particle distribution functions that may differ appreciably from the equilibrium functions. Under these circumstances the question arises as to the stability of such distributions with respect to small oscillations in the field and charge density within the plasma. The present paper is concerned with an analysis of the stability of the electron distribution function with respect to plasma oscillations.

The nature of the field oscillations created when a plasma is disturbed depends, of course, on the unperturbed electron distribution function. For example, as Landau has shown,[1] in the case of a Maxwell distribution the field oscillations will be damped. But if a delta-function like term corresponding to the passage of an electron beam through the plasma is added to the Maxwell distribution, then the field oscillations and the perturbation in the original distribution function increase in time for any velocity of the beam, i.e., the plasma plus beam system is unstable (Akhiezer and Faĭnberg[2]).

More general statements are also true, namely, that any distribution function, which is an arbitrary even function of the velocity with a single maximum at zero is stable, while the superposition of an arbitrary even function and a delta-like term is unstable.[3,4]

2. To establish the general conditions for the stability of the electron distribution function one need only study the behavior of the individual spatial Fourier components of both the potential φ and the perturbation f in the original electron distribution function $F_0(\mathbf{v})$ ($\mathbf{v}$ being the electron velocity). The behavior of these functions as $t \to \infty$ (t being the time) is determined by the singular points of their Laplace transforms φ_p and f_p (here $p = i\omega$ and ω is the complex frequency).

First we shall consider a free plasma, i.e., one not under the influence of external fields. In this case φ_p and f_p are related by[1]

$$f_p(u) = (p + iku)^{-1}\ \{g(u) + ikem^{-1}\varphi_p f_0'(u)\}, \qquad (1)$$

where u is the electron velocity parallel to the wave vector $\mathbf{k}$, $f_0(u)$ is the initial distribution function for u, i.e., $f_0(u) = \int F_0(\mathbf{v})\,d\mathbf{v}_\perp$ ($\mathbf{v}_\perp$ being the electron velocity perpendicular to $\mathbf{k}$), and g is the initial value for $f(u, t)$.

The poles of φ_p are the roots of the equation[1]

$$\frac{i\omega_0^2}{k}\int\limits_C \frac{f_0'(u)\,du}{p + iku} = 1, \qquad (2)$$

where ω_0 is the plasma frequency and the integration is along a line parallel to the real axis and below all the singularities of the integrand.

If all the roots of this equation (with respect to p) lie in the left half-plane, then $\varphi(t)$ will tend to zero as $t \to \infty$. On the other hand, the distribution function will undergo undamped fluctuations of constant amplitude with a frequency ku, which depend on the electron velocity (this behavior of the distribution function is due to the fact that $f_p(u)$ has a purely imaginary pole $p = -iku$ [see Eq. (1)]. On the other hand, if at least one of the roots of Eq. (2) lies in the right half-plane, then $\varphi(t)$ and $f(u, t)$ will increase indefinitely with time and the initial distribution $f_0(u)$ will be unstable.

Thus, the necessary and sufficient condition for the stability of the distribution function $F_0(\mathbf{v})$ consists in there being no roots to the equation

$$G(s) \equiv \int\limits_{-\infty}^{\infty} \frac{f_0'(u)du}{u - s} = \frac{k^2}{\omega_0^2}, \quad s = \frac{ip}{k} \qquad (3)$$

in the upper half-plane s for arbitrary k $(k > 0)$. If no such roots exist, then the function $w = G(s)$, which we shall assume to be finite, maps the upper half plane of s into an area D which does not intersect the positive half-axis of w. This area is bounded by some curve K traced by the point w as s moves along the real axis. Thus this curve is situated in a finite part of the w plane and, because of the principle of maximum modulus, encloses an area D. Therefore, the curve K does not intersect the positive half-axis.

In the case where Eq. (3) does have roots in the upper half plane, then the area D and with it the curve K intersect the positive half-axis.

This intersection means that for some real value of s the value of

$$w = \int_{-\infty}^{\infty} \frac{f_0'(u)du}{u-s} + \pi i f_0'(s) \tag{4}$$

is positive, i.e.,

$$f_0'(s) = 0, \qquad \int_{-\infty}^{\infty} \frac{f_0'(u)du}{u-s} > 0 .$$

The first of these conditions is fulfilled at the extrema of $f_0(u)$, i.e., at $s = u_1, u_2, \ldots, u_n$. Therefore, for the distribution function to be unstable it is necessary and sufficient that at least one of the following inequalities be satisfied*

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\,du}{u-u_j} > 0, \quad j = 1,\ 2,\ldots,\ n\ . \tag{5}$$

It is easy to see that this instability criterion need be verified only for the minimum points of $f_0(u)$. Actually, when s moves along the real axis from $-\infty$ to $+\infty$, the upper half-plane of s remains on the left. Therefore, when w moves along curve K, the area D also remains on the left. This means that w either does not intersect the positive semi-axis at all, or intersects it at least once while passing from the lower to the upper half plane. The derivative $f_0'(s)$ passes from negative to positive values at the same time, which is possible only in the vicinity of the minimum for $f_0(u)$.

Therefore, the stability criterion for $f_0(u)$ has the form

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\,du}{u-u_j} < 0, \qquad f_0'(u_j) = 0, \qquad f_0''(u_j) > 0. \tag{6}$$

3. It follows directly from Eq. (6) that a distribution function with only one maximum is stable.†

If the distribution function $f_0(u)$ has two maxima, one of which is fairly pronounced, it will be unstable. (This is the case when an electron beam with a small velocity dispersion passes through a plasma.)

To prove this we separate the distribution function into two terms, $f_0(u) = f_1(u) + f_2(u)$, with $f_1(u)$ coinciding with $f_0(u)$ outside the interval (u_2, u_2') and equal to $f_0(u_2)$ inside this interval [u_2 is the minimum of $f_0(u)$ lying between the maxima u_1 and u_3 and u_2' is the point nearest u_2 at which $f_0(u_2') = f_0(u_2)$]. It is evident that the contribution of $f_2(u)$ to the integral in (5) will be positive and rather large provided that u_2' is close to u_2. In fact, integration by parts yields

$$\int_{-\infty}^{\infty} \frac{f_2'(u)\,du}{u-u_2} > \frac{\nu}{(u_2'-u_2)^2}, \qquad \nu = \int_{u_2}^{u_2'} f_2(u)\,du > 0 .$$

Therefore, the integral given by (5) will be positive at the minimum point $u = u_2$ if $u_2' - u_2$ is sufficiently small. Consequently, $f_0(u)$ is unstable.

4. We now demonstrate that an arbitrary, spherically symmetrical distribution function $F_0(|\mathbf{v}|)$, which never becomes zero is stable.*

Noting that

$$f_0(u) = \int F_0(\mathbf{v})\,d\mathbf{v}_\perp = 2\pi \int_0^{\infty} F_0\left(\sqrt{u^2+v^2_\perp}\right) v_\perp dv_\perp ,$$

we have

$$f_0'(u) = -2\pi\, u F_0(|u|) .$$

Therefore, Eq. (3) assumes the form

$$2\pi \int_{-\infty}^{\infty} \frac{uF_0(|u|)}{s-u}\,du = \frac{k^2}{\omega_0^2}, \tag{7}$$

and hence

$$2\pi \int_{-\infty}^{\infty} \frac{uF_0(|u|)}{s-u}\,du - 2\pi^2 isF_0(|s|) = \frac{k^2}{\omega_0^2} . \tag{8}$$

The imaginary part of this expression becomes zero when $s = 0$. Therefore, the stability condition requires that

$$-\int_{-\infty}^{\infty} F_0(|u|)\,du < 0,$$

which is always satisfied.

5. We now estimate the growth rate of the oscillations, i.e., the magnitude of $\mathrm{Im}\,\omega$ for the unstable distribution function, assuming that this quantity is sufficiently small. In this connection it is obvious that the curve K intersects the real half-axis of the plane $w \equiv G(s)$ at two neighboring points k_0^2/ω_0^2

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*This criterion has been established independently by Noerdlinger,[5] Penrose,[6] and ourselves.

†This stability condition was obtained by Auer.[7]

*Here the distribution function can have an arbitrary number of maxima and minima. The stability of a spherically symmetrical distribution function with a single maximum has been demonstrated by Penrose.[6]

and k_1^2/ω_0^2 and that $G(s)$ can be expanded in a Taylor series in the vicinity of $s = s_0$ [where $G(s_0) = k_0^2/\omega_0^2$ and $f_0'(s_0) = 0$]. Thus we obtain, correct to quadratic terms,

$$(s - s_0)^2 + \frac{2G'(s_0)}{G''(s_0)}(s - s_0) - \frac{2(k^2 - k_0^2)}{\omega_0^2 G''(s_0)} = 0,$$

and consequently the maximum value of $\operatorname{Im}\omega = k \operatorname{Im} s$ is

$$\max \operatorname{Im}\omega = \max_{k_0 \leqslant k \leqslant k_1} k \operatorname{Im}\left\{-\frac{G'(s_0)}{G''(s_0)} \pm \left(\left[\frac{G'(s_0)}{G''(s_0)}\right]^2 + \frac{2(k^2 - k_0^2)}{\omega_0^2 G''(s_0)}\right)^{1/2}\right\}. \tag{9}$$

In deriving the instability criterion for the distribution function we have disregarded collisions among particles. Obviously these collisions hinder the development of the instability. Hence, the e-folding frequency of the oscillations, $\operatorname{Im}\omega$, must exceed a minimum value equal to the effective collision frequency in order for the instability actually to occur.

6. One can raise the question as to a general form for all the stable $f_0(u)$ distribution functions. Taken together, these functions allow the following representation. If $g(\xi)$ is the Fourier component of $f_0'(u)$, i.e.,

$$f_0'(u) = \int_{-\infty}^{\infty} g(\xi)\, e^{i\xi u}\, d\xi,$$

and if the distribution function is stable, then $g(\xi)$ can be represented in the form

$$g(\xi) = -\int_0^{\xi} \psi(\xi - \xi')\, \psi(\xi')\, d\xi', \tag{10}$$

where

$$\psi(\xi) = \int_{-\infty}^{\infty} e^{-i\xi\lambda}\, d\sigma(\lambda)$$

and $\sigma(\lambda)$ is an arbitrary, continuous, non-decreasing, bounded function.* For every such function there corresponds a definite stable distribution function.

7. We now proceed to examine the conditions for the stability of the distribution function for electrons in a plasma immersed in constant, uniform magnetic field **H**. Here we confine our investigation to plasma waves for which the electric field is longitudinal. The dispersion equation for these waves has the form[8]

$$\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + 2\varepsilon_{13}\cos\theta\sin\theta = 0, \tag{11}$$

where $\epsilon_{ik}(\omega, k)$ is the dielectric permittivity tensor for a plasma in a magnetic field and θ is the angle between **k** and **H**. These quantities are complicated functionals of the distribution function $F_0(\mathbf{v})$ (see Kitsenko and Stepanov[9]).

We assume that the plasma waves are long enough and the magnetic field H is strong enough for the inequality $k\bar{v}_\perp/\omega_H \ll 1$ to hold, here $\bar{v}_\perp$ is the mean value of the electron velocity perpendicular to **H** and $\omega_H = eH/mc$ is the electron gyrofrequency. In this case (11) assumes the form

$$1 - \frac{\omega_0^2\cos^2\theta}{\chi}\int_{-\infty}^{\infty}\frac{f_0'(u)du}{\chi u - \omega} + \frac{\omega_0^2\sin^2\theta}{2\omega_H} \times \int_{-\infty}^{\infty}\left(\frac{1}{\chi u - \omega + \omega_H} - \frac{1}{\chi u - \omega - \omega_H}\right) f_0(u)\, du = 0, \tag{12}$$

where $f_0(u)$ is the distribution of electron velocities parallel to **H** and $\chi = |k\cos\theta|$ is the component of **k** parallel to **H**.

Integrating Eq. (12) by parts one obtains

$$G_H(s) \equiv \int_{-\infty}^{\infty}\left(\frac{\cos^2\theta}{u - s} + \frac{\sin^2\theta}{2s_H}\ln\frac{u - s + s_H}{u - s - s_H}\right) f_0'(u)\, du = \frac{\chi^2}{\omega_0^2}, \tag{13}$$

where

$$s_H = |\omega_H|/\chi.$$

The necessary and sufficient condition for the stability of the $f_0(u)$ distribution function is that the roots of Eq. (13) do not lie in the upper half-plane. Otherwise, as was shown in Sec. 2, the function $w \equiv G(s)$ maps the real axis of s into a curve K that intersects the positive half-axis (here the passage along curve K from the lower to the upper half-plane corresponds to an increase in s).

If s is real, then the real and imaginary parts of $G_H(s)$ are

$$\operatorname{Re} G_H(s) = \cos^2\theta \mathop{\int\!\!\!\!\!\!-}_{-\infty}^{\infty}\frac{f_0'(u)\, du}{u - s} + \frac{\sin^2\theta}{2s_H}\int_{-\infty}^{\infty} f_0'(u)\ln\left|\frac{u - s + s_H}{u - s - s_H}\right| du,$$

$$\operatorname{Im} G_H(s) = \pi\cos^2\theta f_0'(s) + \pi\frac{\sin^2\theta}{2s_H}\int_{s - s_H}^{s + s_H} f_0'(u)\, du.$$

The distribution function will be stable if the real part of $G_H(s)$ is negative for all the values of s for which $\operatorname{Im} G_H(s) = 0$, i.e.,

$$f_0'(u_j)\cos^2\theta + (2s_H)^{-1}[f_0(u_j + s_H) - f_0(u_j - s_H)]\sin^2\theta = 0,$$

$$\cos^2\theta \mathop{\int\!\!\!\!\!\!-}_{-\infty}^{\infty}\frac{f_0'(u)\, du}{u - u_j} + \frac{\sin^2\theta}{2s_H}\int_{-\infty}^{\infty} f_0'(u)\ln\left|\frac{u - s + s_H}{u - s - s_H}\right| du < 0. \tag{14}$$

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*The representation given by (10) was found by B. Ya. Levin (see reference 3).

We note that when this criterion is used, as was explained in Sec. 2, one need take into account only the roots of Eq. (14) for which

$$\int_{u_j}^{s} \operatorname{Im} G_H(s)\, ds = \frac{\pi \sin^2\theta}{2s_H} \int_{s-s_H}^{s+s_H} f_0(u)\, du$$

$$+ \pi \cos^2\theta\, [f_0(s) - f_0(u_j)]$$

has a minimum.

When $\theta \to 0$ and $s_H \to 0$ or $s_H \to 0$, the instability condition (14) coincides with the instability condition in the absence of a magnetic field. Therefore, generally speaking, a magnetic field restricts the class of stable functions.

8. We now show that an even distribution function $f_0(u)$ with a single maximum (at $u = 0$) is stable. In this case Eq. (14) has a single root $u_1 = 0$. Actually, for $u_1 > 0$ the inequality

$$f_0'(u_1) < 0, \quad |u_1 + s_H| > |u_1 - s_H|,$$

is satisfied so that $f_0(u_1 + s_H) < f_0(u_1 - s_H)$, and Eq. (14) can have no solution for $u_1 > 0$. Because $f_0(u)$ is even, this statement holds true even when $u_1 < 0$. The stability condition now assumes the form

$$\cos^2\theta \int_{-\infty}^{\infty} \frac{f_0''(u)}{u}\, du + \frac{\sin^2\theta}{2s_H} \int_{-\infty}^{\infty} f_0'(u) \ln\left|\frac{u + s_H}{u - s_H}\right| du < 0.$$

Obviously this inequality is satisfied, since the integrands are negative for arbitrary u.

9. In conclusion we derive the stability condition for a plasma located in a constant, uniform electric field $\mathbf{E}_0$. Just as in the case of the magnetic field, we confine ourselves to longitudinal plasma oscillations and assume further that $\mathbf{E}_0$ is sufficiently weak.

Beginning with the equations

$$\frac{\partial F}{\partial t} + \mathbf{v}\,\mathrm{grad}\, F + \frac{e}{m}(\mathbf{E}_0 + \mathbf{E}) \frac{\partial F}{\partial \mathbf{v}} + J\{E\} = 0, \qquad (15)$$

$$\mathrm{div}\, \mathbf{E} = 4\pi e n_0 \left(\int F\, d\mathbf{v} - 1\right), \qquad (16)$$

where n_0 is the ion density, $J\{E\}$ the collision integral, and $\mathbf{E}$ the electric field due to plasma oscillations, one can derive[10] the following dispersion equation for high-frequency plasma oscillations for a sufficiently weak field $\mathbf{E}_0$

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\, du}{u - s} + \frac{ieE_0}{mk} \cos\theta \int_{-\infty}^{\infty} \frac{du}{u - s} \frac{d}{du}\left[\frac{f_0'(u)}{u - s}\right] = \frac{k^2}{\omega_0^2}, \qquad (17)$$

where $f_0(u)$ is the initial electron distribution function and θ is the angle between $\mathbf{k}$ and $\mathbf{E}_0$. In deriving this equation we have neglected the collision integral, which, however, is implicitly contained in f_0, whose form depends on $J\{f_0\}$.

Integrating the second term in Eq. (17) by parts, separating the real and imaginary terms and making use of the method explained in Sec. 2, we obtain the following condition for the stability of the distribution function $f_0(u)$

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\, du}{u - u_j} - \frac{\pi e E_0}{4mk} \cos\theta\, f_0'''(u_j) < 0, \qquad (18)$$

where u_j are the roots of the equation

$$f_0'(u_j) + \frac{eE_0}{2\pi mk} \cos\theta \int_{-\infty}^{\infty} \frac{f_0'''(u)}{u - u_j}\, du = 0; \quad e < 0.$$

The authors deem it their pleasant duty to thank K. N. Stepanov and A. B. Kitsenko for their valuable advice and assistance.

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SOVIET PHYSICS JETP VOLUME 13, NUMBER 3 SEPTEMBER, 1961

Letters to the Editor

EFFECT OF ANISOTROPY ON THE PROPERTIES OF SEMICONDUCTORS

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THE theory of superconductivity[1,2] has been developed, as is well known, for an isotropic model of the metal. The present paper is devoted to an investigation of the effect of anisotropy on the properties of superconductors.

The electron-phonon interaction is characterized by the Hamiltonian

$$H' = \sum_{\mathbf{k}'=\mathbf{k}+\mathbf{q}} g\,(\omega\,/\,2V)^{1/2} a^{+}_{\mathbf{k}'s} a_{\mathbf{k}s} b_{\mathbf{q}} + \text{compl. conj.} \qquad (1)$$

where $a_{\mathbf{k}s}$ is the second-quantized amplitude, and g is the coupling constant, the anisotropy of which must be taken into account when solving the problem under consideration.

We start with an expression for the matrix element that characterizes the electron transitions in the metal due to the interaction between the electrons and the lattice (we disregard Umklapp processes)[3]

$$K = -\frac{h^2 i}{m}\left(\mathbf{k}-\mathbf{k}',\ \int \operatorname{grad} u_{\mathbf{k}} \frac{\partial \overset{*}{u}_{\mathbf{k}'}}{\partial s}\, d\tau_0\right)$$
$$-\left[E_{\mathbf{k}} - E_{\mathbf{k}'} - \frac{h^2}{2m}(k^2 - k'^2)\right]\int u_{\mathbf{k}} \frac{\partial \overset{*}{u}_{\mathbf{k}'}}{ds}\, d\tau_0. \qquad (2)$$

The Coulomb interaction is not considered here. On the basis of symmetry considerations, we can determine the form of the integrals that enter in (2). We then obtain from (2)

$$g_j = g_{0j}\left\{b\,\frac{\mathbf{k}+\mathbf{k}'}{q}\,\mathbf{e}_{\mathbf{q}j}\left[E_{\mathbf{k}} - E_{\mathbf{k}'} - \frac{1}{2m}(k^2 - k'^2)\right]\right.$$
$$\left. + a\,\frac{\mathbf{q}}{q}\,\mathbf{e}_{\mathbf{q}j} + c\left(\mathbf{k}+\mathbf{k}',\ \frac{\mathbf{q}}{q}\right)(\mathbf{k}+\mathbf{k}',\ \mathbf{e}_{\mathbf{q}j})\right\}, \qquad (3)$$

where e_{qj} is the unit vector of phonon polarization $(j = 1, 2, 3)$, $q = |\mathbf{k}' - \mathbf{k}|$, and a, b, and c are slowly-varying functions of q, which we assume constant in a layer $h\omega_0$ near the Fermi surface. We neglect terms that contain the unit vectors of the lattice, which in the case of s electrons are smaller than the terms written out here.

Carrying out in the Hamiltonian (1) a canonical transformation after Bogolyubov,[2] we can obtain an equation for the gap in the energy spectrum

$$\Delta(\mathbf{k}) = \frac{1}{2(2\pi)^3}$$
$$\times \sum_{j=1}^{3} \int g_j^2(\mathbf{k},\mathbf{k}')\,\frac{\omega_j(q)}{[\omega_j(q) + \tilde{\varepsilon} + \tilde{\varepsilon}']}\,\frac{\Delta(\mathbf{k}')\,d\mathbf{k}'}{\sqrt{\Delta^2(\mathbf{k}') + \xi^2(\mathbf{k}')}}, \qquad (4)$$

where g_j is determined from (3). The substitution $\omega_j(\omega_j + \tilde{\epsilon} + \tilde{\epsilon}')^{-1} \approx 1$ leads to the renormalization of g_{0j}.[2]

We first consider the case of a closed Fermi surface with a dispersion law

$$E = k_z^2/2m_1 + (k_x^2 + k_y^2)/2m_2$$

(ellipsoid of revolution). We assume that $m_1 - m_2 = \Delta m \ll m_1$ and $h\omega_0/E_F \ll m_1/\Delta m$, and take account of the fact that $b < a$ and $c < a$. We then solve Eq. (4) and find (for $T = 0$) $\Delta(\vartheta) = \Delta_0 + \Delta_1$ where

$$\Delta_1 = \frac{b+c}{a}\left(\frac{\Delta m}{m_1}\right)^2 \Delta_0\left(\frac{11}{6}\cos^4\vartheta - \cos^2\vartheta + \frac{1}{2}\right), \qquad (5)$$

and Δ_0 is determined by

$$1 = 4\pi g_{01}^2 a^2 \alpha\left[\left(1 - \frac{\Delta m}{m_1}\right)^{-1/2} + \left(\frac{\Delta m}{m_1}\right)^2 \frac{b+c}{2a}\right]\ln\frac{2\tilde{\omega}}{\Delta_0},$$
$$\alpha = m_2(2m_2E_F)^{1/2}$$

(ϑ is the angle between the z axis and the vector **k**).

Calculation for $T \to T_c$ leads to the following result:

$$\Delta(\mathbf{k}, T) = \Delta_1(T)\left[1 + \left(\frac{\Delta m}{m_1}\right)^2 \frac{b+c}{a}\left(\frac{11}{6}\cos^4\vartheta - \cos^2\vartheta + \frac{1}{2}\right)\right], \qquad (6)$$

where $\Delta_1(T)$ is given by

$$1 = 4\pi g_{01}^2 a^2 \alpha\left(1 - \frac{\Delta m}{m_1}\right)^{-1/2}\left[\ln\frac{2\omega\gamma}{\pi T}\left(1 + \left(\frac{\Delta m}{m_1}\right)^2 \frac{b+c}{2a}\right)\right.$$
$$\left. - \frac{7}{8\pi^2}\zeta(3)\frac{\Delta_1^2(T)}{T^2}\right], \qquad (6')$$

or

$$\frac{\Delta_1(T)}{T} = 3.06\left[\left(1 - \frac{T}{T_{\text{к}}}\right)\left(1 + \left(\frac{\Delta m}{m_1}\right)^2 \frac{b+c}{2a}\right]^{1/2},$$

where T_c is determined from (6′) $[\Delta_1(T_c) = 0]$.

We consider next the case of the simplest open Fermi surface — cylindrical surface with dispersion law $E = k_\perp^2/2m$. The calculation of $\Delta(\mathbf{k})$ leads to the following result:

$$\Delta = \Delta_0 + \Delta_1(k_z) + \Delta_1(-k_z),$$
$$\Delta_1(k_z) = \frac{b+c}{a} g_{01}^2 \Delta_0\left[\frac{1}{3}s^3 + (2k_z^2 - 2k_z k_{z\,\max} - 4k_{\perp F}^2)\,s\right.$$
$$\left. + 8k_z k_{\perp F}^2 \ln\frac{k_{z\,max} + k_z + s}{k_{\perp F}} + 5k_{\perp F}^3 - 8k_z^2 k_{\perp F}\right],$$
$$s(k_z) = [4k_{\perp F}^2 + (k_{z\,\max} + k_z)^2]^{1/2}. \qquad (7)$$

We now calculate the expression for the heat capacity of the superconductors in the cases under consideration. As a result we obtain the following expression for the ellipsoid:

When $T = T_c$

$$\frac{C_s(T_c)}{C_n(T_c)} = 2{,}4 + 1{,}4\,\frac{b+c}{a}\left(\frac{\Delta m}{m_1}\right)^2. \tag{8}$$

In the low-temperature region we obtain

$$\frac{C_s(T)}{C_n(T_c)} = \frac{1}{\pi^2 T_c}\left(\frac{\pi}{2}\right)^{1/2} T^{-3/2}\Delta^{5/2}(0)\exp\left(\frac{-\Delta(0)+\beta}{T}\right)\sqrt{\frac{\pi T}{2\beta}},$$

$$\Delta(0) = \overline{\Delta(\vartheta)}, \qquad \beta = \frac{b+c}{a}\left(\frac{\Delta m}{m_1}\right)^2\Delta(0). \tag{9}$$

The formula is applicable at temperatures that satisfy the condition

$$T/\Delta(0) \ll (\Delta m/m_1)^2 (b+c)/a.$$

Analogous results are obtained also for a cylinder. However, in the absence of a small parameter similar to $\Delta m/m$ (both for an ellipsoid and for a cylinder), the variation of the gap and of the heat capacity near T_c may be more complicated.

It is easy to see that as $T \rightarrow 0$ the heat capacity C_s is determined by the smallest gap Δ_{min} (see also reference 4); on the other hand, near T_c it is determined by the average value of the gap $\Delta(0)$. Therefore C_s diminishes in the anisotropic model more slowly with decreasing temperature than in the isotropic model. This agrees with the experimental data (see references 4 and 5). Since the expression for C_s (and obviously also for the critical magnetic field H_c) contains the parameters of the Fermi surface (m_1 and m_2 for an ellipsoid, $k_{\perp F}$ and $k_{z\,max}$ for a cylinder), C_s and H_c are no longer universal functions of the temperature, as in the isotropic model, and this explains the difference between the experimental curves of C_s and H_c for different superconductors.

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MEASUREMENT OF ANGULAR DISTRIBUTIONS IN THE REACTION $Al^{27}(p, p')Al^{27}$ *AT 6.6 Mev WITH A MAGNETIC ANALYZER*

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It is well known that the study of angular distributions in inelastic proton scattering is an important source of information about the properties of nuclear states and the mechanism of inelastic scattering. In such investigations, it is necessary to measure angular distributions of groups of particles with nearly the same energy; this can be done quite reliably by magnetic analysis. On the other hand, the use of a magnetic analyzer leads to a significant increase in the difficulty of the experiment. Therefore, in the overwhelming majority of experiments, the angular distributions are measured with scintillation spectrometers, nuclear emulsions, etc.[1] Because of their low resolution, and because of overloading due to elastic protons, these techniques can measure angular distributions only for one or two excited states of the nucleus.

The ease of field measurement[3] and regulation and the relatively high intensity of the magnetic analyzer described previously[2] enabled us to measure angular distributions for the six groups of elastic and inelastic protons from Al^{27} leading to the excited states at 0.840, 1.014, 2.216, 2.743, and 3.000 Mev (the last level is, according to reference 4, a doublet: 2.976 and 2.999 Mev) without excessive increase in the difficulty of the experiment.

The protons were accelerated in the 120-cm cyclotron of the Moscow State University. A monochromatic beam was obtained by placing collimating slits after the focusing magnet. The full energy spread of the beam was about 45 kev.

The spectrometer for analyzing the secondary particles was set on a rotating support. The particle detector was a ZnS scintillation counter with an FÉU-29 photomultiplier.

The measured differential cross sections (in mb/sr) for elastic and inelastic proton scattering on Al^{27} at 6.6 Mev are shown in the table. The accuracy of the cross sections is about 10%.

Q, Mev	Scattering angle in lab system									l	R, f
	32°49′	48°28′	63°04′	76°54′	90°24′	109°10′	124°13′	138°17′	151°57′		
0.000	850.20	302.30	177.00	109.20	60.50	38.50	44.50	65.00	64.20	—	—
—0.840	4.75	7.87	7.85	7.47	6.55	4.35	—	2.52	2.0	2	5.6
—1.014	6.93	9.55	10.30	10.10	8.88	7.80	7.00	6.34	—	2	5.6
—2.216	10.40	12.95	14.30	16.00	17.70	15.35	10.60	9.75	8.15	3	5.6
2.743	8.45	11.20	12.55	13.35	13.05	12.30	11.60	10.50	7.36	2	5.0
3.000	12.20	14.60	17.80	19.90	20.50	16.70	14.05	10.20	9.75	2	5.0

The angular distributions of all the particle groups (in the center-of-mass system) have the characteristic features peculiar to direct interaction angular distributions. The total scattering cross section for the six levels is about 700 mb, which is larger than the cross section for compound nucleus formation (about 600 mb). This leads to the conclusion that direct interaction makes a substantial contribution in 6.6-Mev scattering. The orbital angular momenta, l, transferred to the nucleus, determined by comparison with direct interaction theory,[5] are given in the table together with the corresponding values of the interaction radius R (in 10^{-13} cm).

In conclusion, the authors express their gratitude to Z. F. Kalacheva, I. V. Kretov, G. S. Tyurikov, N. S. Kirpichev, and M. Kh. Listov for aid in this work, and also to the crew of the cyclotron, particularly Yu. A. Vorob'ev, A. A. Danilov, V. P. Khlapov, and E. Kir'yanov, for assuring proper operation of the accelerator.

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A METHOD FOR THE MEASUREMENT OF PHOTOPRODUCTION OF π^+ MESONS ON HYDROGEN CLOSE TO THRESHOLD

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WHEN combining the experimental data on the photoproduction of π^+ mesons close to threshold[1] one finds that the square of the matrix element increases as the photon energy decreases. However, very close to threshold the accuracy of the measurements is not good enough to draw definite conclusions.[2]

The absence of good data on the photoproduction cross section close to threshold is due to the large experimental difficulties. Very close to threshold (152 — 160 Mev) one cannot use liquid or gas targets because of the energy loss of the mesons in the target and because of the large background. In this energy region one has to use thin polyethylene targets.[3] However, one then has to perform $CH_2 - C$ difference experiments, which for large yields from carbon decreases the statistical accuracy of the results.

We shall describe a method which allows to circumvent these difficulties and we shall give results of an experimental demonstration of its feasibility. The method is based on the kinematical difference between photoproduction on hydrogen and on carbon, in particular on the difference in the thresholds. The following reactions have the lowest thresholds:

$$\gamma + {}_6C^{12} \rightarrow \pi^+ + {}_5B^{12}, \tag{1}$$

$$\gamma + {}_6C^{12} \rightarrow \pi^+ + n + {}_5B^{11} \tag{2}$$

$$\gamma + {}_6C^{12} \rightarrow \pi^+ + n + n + {}_5B^{10}. \tag{3}$$

The thresholds for these reactions are respectively

154.5, 157.7, and 169.1 Mev. The π^+ production threshold on hydrogen is 151.3 Mev.

In Fig. 1 the photon energy has been plotted as a function of the meson energy for different directions of the outgoing π^+ meson both for production on hydrogen and by reaction (1). As can be seen from the thresholds the curves for reactions (2) and (3) will be still higher.

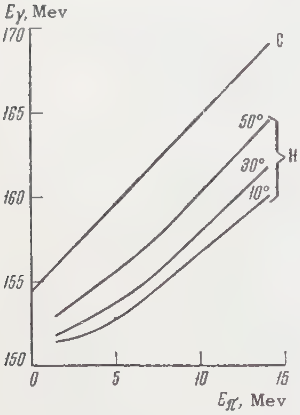

FIG. 1. Dependence of the photon energy on the meson energy at different angles of the outgoing meson.

From Fig. 1 one sees that for a given meson energy and a small angle one can choose such a maximum photon energy in the bremsstrahlung spectrum, $E_{\gamma\,\max}$, that no photoproduction from carbon is possible. For example, for $E_{\gamma\,\max} = 158$ Mev mesons with 3 Mev at an angle 30° will have been produced exclusively on hydrogen by photons with an energy 153 Mev. However, under these conditions only the very tip of the bremsstrahlung spectrum will be utilized. One therefore should increase the photon energy $E_{\gamma\,\max}$ as far as possible to increase the yield from hydrogen while keeping the background of mesons from carbon sufficiently low. Fortunately the photoproduction from nuclei is a single nucleon process.[4,5] Thus reaction (1) is a partial and unlikely process of reaction (2). Accordingly quite a high energy $E_{\gamma\,\max}$ can be chosen.

In order to verify these considerations an experiment has been performed. The apparatus is shown schematically in Fig. 2.

A polyethylene target of 0.0173 g/cm^2 was placed in the photon beam of the synchrotron of the Physics Institute of the Academy of Sciences. The energy was $E_{\gamma\,\max} = 175$ Mev. Mesons of energy 0.5 to 5.5 Mev emitted at 30° in the lab system* were counted. Nuclear NIKFI-K emulsions 400μ thick were used as the detector. Target and emulsions were placed in a vacuum chamber. The energy loss for mesons of average energy in the target was ~ 0.1 Mev.

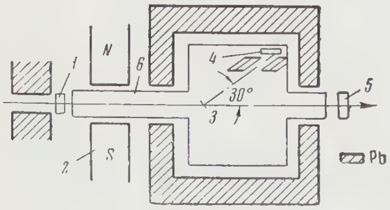

FIG. 2. Experimental setup. 1, 5 – ionization chambers, 2 – magnet, 3 – target, 4 – nuclear emulsion, 6 – vacuum chamber.

In such an experimental arrangement in the nuclear emulsions both positive mesons from hydrogen and carbon and negative mesons from carbon will be seen. The π^- mesons are created in carbon in reactions similar to those of the π^+ mesons, namely:

$$\gamma + {}_6C^{12} \rightarrow \pi^- + {}_7N^{12}, \tag{1a}$$

$$\gamma + {}_6C^{12} \rightarrow \pi^- + p + {}_6C^{11}. \tag{2a}$$

The thresholds for these reactions are 157.8 and 158.9 Mev respectively.

The contribution of π^+ mesons from carbon can be easily estimated from the number of observed π^- mesons since the π^+/π^- photoproduction ratio in carbon is known for the energy range of interest[6] and equals 2.7 ± 0.4.† This allows one to avoid the necessity of performing the rather tedious work of determining the yield of π^+ mesons from carbon.

The experimental results are:

$E_{\gamma\,max}$, Mev:	175	250
$\pi^-(CH_2)/\pi^+(CH_2)$:	0.22±0.06	0.55±0.14
$\pi^+(CH_2)/\pi^+(H_2)$:	1.09±0.03	1.26±0.05

It follows from these data that the relative yield of π^+ mesons from carbon is (9 ± 3)% of that from hydrogen. For comparison purposes the analogous quantities are given also for $E_{\gamma\,\max} = 250$ Mev. Thus one sees clearly that the proposed method allows to decrease the π^+-meson background from the carbon in the polyethylene target considerably and to increase the statistical accuracy of the experimental results obtained in a given running time.

In addition, our experimental data allow a determination of the most probable type of the reaction.

We now shall compare the experimental ratio $\pi^-(C)/\pi^+(H_2)$ with a calculation for $E_{\gamma\,\max} = 250$ Mev which is based on the assumption that the total meson yield from carbon is due to the reactions (1) and (2). In the determination of this ratio for reaction (2) the distribution of nucleon momenta in the nucleus was taken into account.[7] We obtain:

	Experiment	Calculation	
		Reaction (1)	Reaction (2)
$\pi^-(C)/\pi^+(H_2)$:	0.24 ± 0.07	0.67 ± 0.17	0.34 ± 0.09

One can see from this that the results of the present experiment support the supposition on the one nucleon mechanism of π-meson production in nuclei.

It should be mentioned that the suggested method will be even more effective for the measurement of the photoproduction of π mesons from deuterons close to threshold using heavy polyethylene.

In conclusion the authors express their gratefulness to Professor P. A. Cerenkov for his interest in the work.

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*At the energy $E_{\gamma\max} = 175$ Mev mesons at this angle can be produced only in reactions (1) and (2).

†Comparing the thresholds for π^+ and π^- production one can easily see that the ratio π^-/π^+ at $E_{\gamma\max} = 175$ Mev is practically the same as that found at $E_{\gamma\max} = 250$ Mev.[6]

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A NOTE ON THE ANISOTROPY IN THE ANGULAR DISTRIBUTION OF PARTICLES FROM NUCLEAR INTERACTIONS AT 10^{12} ev.

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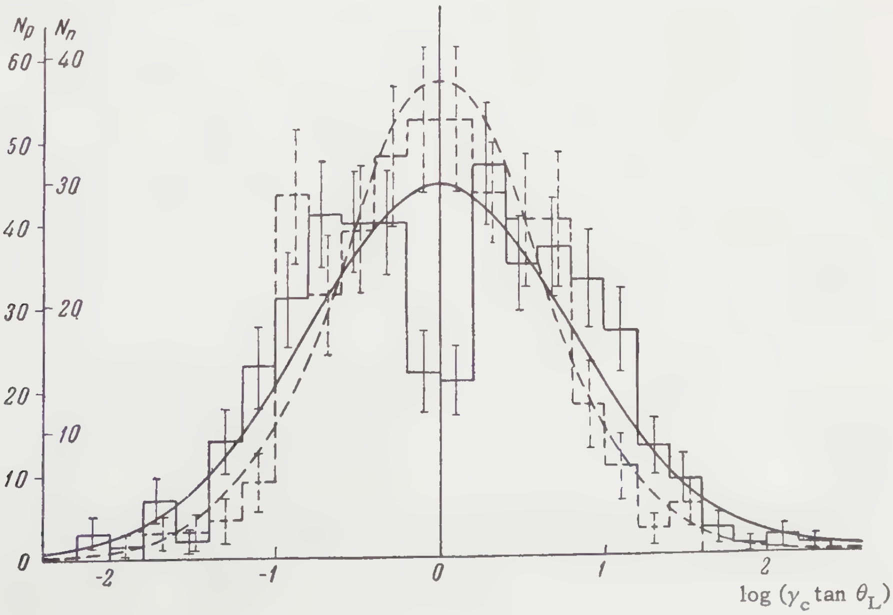
In the work of Gierula et al.[1] it was shown that the angular distribution of secondary particles from nuclear interactions with primary energy $> 10^{12}$ ev has a significant anisotropy and that besides the anisotropy, characterized by $\sigma > 0.6$, there is a structure with two maxima. This result was confirmed in reference 2.*

We would like to point out that the interactions produced by charged particles (p) have, on the average, a greater anisotropy than those with neutral (n) primaries. We show in the figure the distribution of values of $\log(\gamma_c \tan\theta_L)$ separately for the groups with p primaries (solid curve) and n primaries (dashed curve). The first group includes 19 cases with $\langle\gamma_c\rangle = 54$, predominantly from the literature. In the n interactions there are 13 cases with $\langle\gamma_c\rangle = 44$. We determined the anisotropy with the aid of σ as in reference 1; for the group p we have $\sigma = 0.82$, and for the n distribution we obtained $\sigma = 0.64$.



A similar result was obtained for lower energy interactions investigated in our laboratory. A group of 45 p interactions with $\langle\gamma_c\rangle = 8$ has $\sigma = 0.6$ while a group of 11 n interactions with $\langle\gamma_c\rangle = 8.3$ has $\sigma = 0.5$.

We think that the smaller interaction anisotropy is not due to inaccuracy in the determination of the axis, because the distribution of particles in a wide cone, which is not very sensitive to the choice of axis, reveals this effect even better than that in a narrow cone. Finally, the random character of the choice of individual events from the literature could have affected our results as shown in the figure to some extent, but in the low energy p and n groups, the method of selection has practically no effect; they were obtained by scanning the same volume of emulsion.

We think that the average anisotropy in the p group is increased by the contribution of interactions produced by mesons, while the n group consists of interactions produced only by neutrons. A more detailed analysis will be published in the Czechoslovakian Physics Journal.

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*The authors thank Professor J. Gierula for sending this preprint.

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[2] Gierula, Haskin, and Lohman, preprint.

Translated by M. Bolsterli
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CONSECUTIVE INTERACTIONS OF HEAVY NUCLEI OF THE PRIMARY COSMIC RADIATION

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CONSECUTIVE interactions of heavy nuclei, and interactions in a beam of their several fragmentation products ("parallel interactions") are interactions of particles having an identical energy per nucleon, and their investigation may therefore yield information on possible asymmetries in the angular distribution of the particles produced.

We summarize here preliminary results obtained in a stack of nuclear emulsions irradiated during the 1955 Po-valley expedition. We have, so far, found six pairs of consecutive or parallel interactions of heavy nuclei. Their characteristics are shown in the table. N_h denotes the number of evaporation tracks, Z is the charge of the incident nucleus, Z_i are the charges of the fragmentation products, γ_c is determined from the relation $\log \gamma_c = \log \cot \theta_i$, n_1 and n_2 are the number of particles in the narrow and wide cone, respectively, divided by γ_c, and n_1' and n_2' are the number of particles in the cones divided by γ_c', where $\gamma_c' = (\gamma_1\gamma_2)^{1/2}$.

It should be noted that the values of γ_c in separate interactions of a pair differ considerably in

№	N_h	Z	Z_i	n_s	γ_c	$n_1 : n_2$	γ_c'	$n_1' : n_2'$
			Consecutive interactions					
208 a	0	~15	$Z_1 = 2,\ Z_2 = 4,\ Z_3 = 5$	25	23±5	11 : 9	17	15 : 5
208 b	26	2		32	2.6±0.5	17 : 15	2.5	17 : 15
232 a	14	~6	$Z_1 = Z_2 = 2$	30	1.4±0.3	12 : 10		
232 b	1	2		5	3.7±1.8	2 : 2	3.7	2 : 2
203 a	0	~17	$Z_1 = Z_2 = 2,\quad Z_3 = 12$	15	27±7	7 : 7	27	7 : 7
203 b*	16	~12		115	9.5±1	56 : 56	7	65 : 47
207 a	0	~18	$Z_1 = 2,\quad Z_2 = 13$	9	8.8±3.5	3 : 3	8.8	3 : 3
207 b	4	~13	$Z_1 = Z_2 = 2$	20	4.5±1.3	6 : 5	3.4	6:5
			Parallel interactions					
9	4	~5		47	8.3±1.2	25 : 17	7	26 : 16
227	35	2		38	3.1±0.5	17 : 19	3.2	17 : 19
190	12	2		51	15.5±2.2	22 : 28	18	22 : 28
191	29	~17	$Z_1 = 4,\quad Z_2 = Z_3 = 2$	160	21.6±1.7	80 : 75	17	94 : 61

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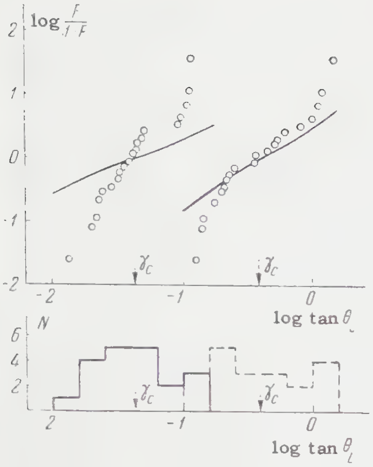
*The angular distribution of particles for the case 203b has been kindly reported to us by Dr. E. Fenyves from Budapest. The events 203a and 203b are described in reference 1.



Angular particle distribution (top: integral; bottom: differential) for the event 208a and b.

the majority of cases. In the case 208a and b (see figure), and cases 9 and 227, the difference is even greater than three times the statistical fluctuations. Since the energy per nucleon in the laboratory system should be identical for both incident nuclei, this difference in the values of γ_c is the consequence of non-identical effective masses of the interacting nuclei M_1 and M_2 if we assume that they interact as one body. In the event 208a and b, and for a simplifying assumption $M_1(a)/M_2(a) = M_2(b)/M_1(b)$, we find that the ratio of the effective masses is equal to the ratio of $\gamma_c(a)/\gamma_c(b) = 9$. This corresponds to a small N_h and a large Z for the case 208a, but to a large N_h and a small Z for the case 208b. However, the ratio of the effective masses in the case under consideration is unusually large, so that it is difficult to use the hydrodynamical model[2] represented by the solid curves in the integral distributions shown in the figure (upper part).

Using the values of γ'_c, we obtain, as a rule, a smaller ratio than when using γ_c, but the forward-backward symmetry in the number of emitted particles ($n'_1 : n'_2$ or $n'_2 : n'_1$ respectively) increases, especially in cases with a large anisotropy. Such an asymmetry is observed in the interactions 208a, 9, 191, and others. Evidently, the number of emitted particles is proportional to the effective masses of the interacting particles.

A more detailed analysis will be published in the Czechoslovak Journal of Physics.

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SOUND ABSORPTION IN ROCHELLE SALT CLOSE TO ITS LOWER CURIE POINT

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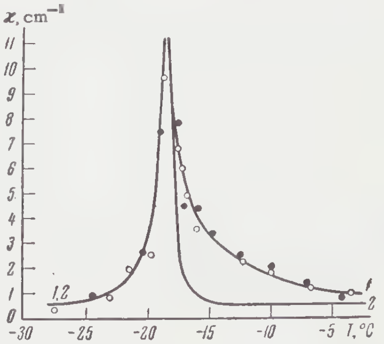
As we have previously reported,[1] there is an anomalously large absorption of transverse elastic waves in Rochelle salt at temperatures close to its upper Curie point ($\Theta = 24°$ C); these waves are propagated along the crystallographic z axis and are polarized along the y axis. (The ferroelectric axis of the crystal is directed along x.) Such a sound absorption is the consequence of an increase in the relaxation time which is characteristic of the process of establishing the state of thermodynamic equilibrium in a crystal which is undergoing a second-order phase transition (see below). With the aim of generalizing the result given in reference 1, we carried out measurements of the absorption of a wave of given polarization and frequency $\nu = \omega/2\pi = 5$ Mc/sec in Rochelle salt near its lower Curie point ($\Theta = -18°$ C). The results of these measurements are given in the drawing by the curve 1 — 1; the temperature T of the crystal is plotted along the abscissa and the values of the amplitude absorption coefficient κ in cm^{-1} along the ordinate.



The theory of this phenomenon was developed by Landau (see reference 2) who showed that*

$$\varkappa^2 = \frac{\omega^2\rho}{2}\left(\mu\left[1 + \frac{8\pi\lambda^2(\mu/\varepsilon + 2\pi\lambda^2)}{\mu^2(\varepsilon^{-2} + 16\pi^2\omega^2\gamma^{-2})}\right]^{1/2} - \mu - \frac{4\pi\lambda^2/\varepsilon}{\varepsilon^{-2} + 16\pi^2\omega^2\gamma^{-2}}\right), \tag{1}$$

where (near the lower Curie point) $\epsilon = \epsilon_x = 4\pi C/(\Theta - T)$ for $T < \Theta$ and $\epsilon = \epsilon_x = 2\pi C/(T - \Theta)$ for $T > \Theta$ is the dielectric constant of the Rochelle salt, ρ is its density, $\mu = \mu_{zyzy}$ is the shear elastic coefficient for constant induction D_x; $\lambda = \lambda_{x,zy}$

is the piezoelectric constant of the crystal, γ is the coefficient in the kinetic equation $\partial D_X/\partial t = \gamma\partial\Phi/\partial D_X$, Φ is the thermodynamic potential of the crystal, which is represented by a function of D_X and the shear stress σ_{zy} (on this decomposition of Φ, see reference 3).

The coefficient γ can be estimated by substituting the known values of all parameters in (1) and one of the values of κ found experimentally. In taking into account the numerical values of the quantities entering into (1), we make use of the approximate formula

$$\varkappa = 8\sqrt{\frac{\rho}{\mu}}\,\frac{\pi^2\lambda^2\omega^2/\gamma}{\varepsilon^{-2} + 16\,\pi^2\omega^2\gamma^{-2}} \qquad (2)$$

for comparison of theory with experiment.

The theoretical curve 2 — 2 corresponding to Eq. (2) is shown in the drawing. In the scale chosen for the drawing, the lower temperature branch of the curve 2 — 2 coincides with the corresponding branch of the experimental curve 1 — 1. The excellent agreement of the Landau theory with experiment for $T < \Theta$ made it possible for us to estimate the relaxation time $\tau = 4\pi\epsilon/\gamma \approx 3.4 \times 10^{-8}/(\Theta - T)$ sec for $T < \Theta$.

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*We note that misprints occurred in Eq. (1) which was given in reference 2 under the number (35), and also in the following Eq. (36).

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HYPOTHESIS OF CONSERVED VECTOR CURRENT AND GLOBAL SYMMETRY OF WEAK INTERACTIONS

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THE existence of divergences among the values of the vector coupling constants in β decay (decay of O^{14}) and μ decay, established on the basis of new experimental data, has been reported in recent papers.[1-6]

In the present communication we call attention to the fact that one of the reasons for the difference between the indicated coupling constants may be the nonconservation of the weak vector current, even without allowance for the radiative corrections for the electromagnetic and the weak[8] interactions.

In reference 7 an analogy was developed between the weak interaction and the electromagnetic one, based on the assumption of global symmetry of weak interactions and on the hypothesis of two forms of neutrinos — electron and muon[9,10] (global symmetry in weak interactions was considered also by Treiman[12]). It was assumed here that the properties of Fermi particles in weak interactions can be obtained from the classification of massless fermions with respect to possible values of charges (electric, baryon, and lepton) and the chirality γ_5, and the mass degeneracy is removed by strong interactions. It is easy to verify that since strong interactions are as a whole not globally symmetrical,[11] the weak vector current obtained in reference 7 is not conserved. We assume that the pion interactions are globally symmetrical and give rise to a mass difference between the leptons and baryons, while K-meson interactions lift the mass degeneracy of the baryon isotopic multiplets. Then the nonconservation of the isovector current will be due only to K interactions. Since the renormalization of even the axial coupling constant is small (0.2), we can expect the renormalization of the vector constant, in which π interactions make no contribution, to be even much smaller. An estimate of this reduction is the ratio of the splitting of the masses of the baryons to the mass difference between baryons and leptons (only the Σ-Λ mass difference is of importance here), equal to ~ 0.1. This gives a difference in constants of about 0.02 in β and μ decay, which does not contradict the available experimental data.[5,6] It must be kept in mind that strong interactions renormalize the constant g_β, while weak ones renormalize g_μ. An account of this fact is essential if information is to be obtained on the intermediate heavy vector meson from an analysis of the radiative corrections based on weak interaction (see reference 8).

It is essential to note that in spite of the theoretical attractiveness of the assumption of the global symmetry of the π interactions, the estimate obtained here for the difference of the β- and μ-decay coupling constants is essentially based on the considerably weaker requirement, namely $g_{\Sigma\Lambda\pi} = g_{\Sigma\Sigma\pi}$.

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FISSION INDUCED BY MUONS

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THERE is an observable probability that the energy released in the 2p to 1s transition in μ-mesonic atoms is converted into nuclear excitation.[1,2] In uranium and plutonium, the excitation energy amounts to ~6.3 Mev. At this energy the nucleus can decay into various channels, namely, by γ emission, neutron emission, and fission.

Grechukhin (private communication) has stated that the fission channel is forbidden. Qualitatively, this effect is due to the fact that the meson in the 1s state near the nucleus hinders the nuclear deformation which would lead to fission. Therefore, the fission barrier is higher with meson present than without the meson. The closer the fission threshold is to the excitation energy, the larger the effect of the presence of the 1s meson is on the fission probability.

Since the rotational energy of the nucleus is much less than the energy of the meson in the ground state, rotational effects and nuclear fission are adiabatic with respect to the mesonic motion. Thus, the calculation of the effect of the meson on fission can be carried out for a fixed orientation of the nuclear axis.

The binding energy of the meson in the ground state decreases when the nucleus is deformed and thus the energy threshold for fission is correspondingly increased, since the potential curve for nuclear fission with the meson present is $E^{\mu}_{\text{nuc}} = E^{0}_{\text{nuc}} + E_{\mu}$ where E^{0}_{nuc} is the potential curve for fission without the meson and E_{μ} is the binding energy of the meson which depends on the nuclear deformation parameters. To find E_{μ}, it is necessary to solve the Schrödinger equation for the meson in the Coulomb field of the deformed nucleus. We assume for simplicity that up to the saddle point the nucleus has the form of an ellipsoid of revolution. The Coulomb potential of a uniformly charged ellipsoid of revolution with semi-axes a and b has the form

$$\varphi(\alpha, \beta) = \frac{Ze}{c}\Big\{[1 - P_2(\operatorname{ch}\alpha)P_2(\cos\beta)]\ln\operatorname{cth}\frac{\alpha_0}{2} + \frac{3}{2}\frac{\operatorname{ch}^2\alpha}{\operatorname{ch}\alpha_0}P_2(\cos\beta) + \frac{3}{4}\Big(1 - \frac{\operatorname{sh}^2\alpha}{\operatorname{sh}^2\alpha_0}\Big)\frac{\sin^2\beta}{\operatorname{ch}\alpha_0}\Big\} \quad \text{for } \operatorname{ch}\alpha \leqslant \operatorname{ch}\alpha_0 = \frac{a}{c},$$

$$\varphi(\alpha, \beta) = \frac{Ze}{c}\Big\{[1 - P_2(\operatorname{ch}\alpha)P_2(\cos\beta)]\ln\operatorname{cth}\frac{\alpha}{2} + \frac{3}{2}\operatorname{ch}\alpha P_2(\cos\beta)\Big\} \quad \text{for } \operatorname{ch}\alpha \geqslant \frac{a}{c}, \qquad (1)^*$$

where Ze is the nuclear charge, $c^2 = a^2 - b^2$, $P_2(x)$ is the Legendre polynomial of order two, and α and β are the degenerate ellipsoidal coordinates.

The Schrödinger equation for the ground state of the mesonic atom with the potential (1) was numerically integrated on an electronic computer. Below are presented the values of the binding energy of the meson in the ground state of the U^{238} mesonic atom as a function of the ratio of the semiaxes of the nucleus

a/b =	1.2	1.4	1.6	1.8	2	2.2	2.5
E_{μ} (Mev) =	11.89	11.79	11.66	11.53	11.36	11.21	11.01

With the hydrodynamic model for fission,[3] it is possible to find the ratio of the nuclear semiaxes at the saddle point, $(a/b)_{\text{sp}}$. This ratio and the increase, ΔE, in the height of the fission barrier for several nuclei are given in the table [$(a/b)_0$ is the statistical nuclear deformation,[4] E_{thr} is the photofission threshold[5]].

Nucleus	$(a/b)_0$	$(a/b)_{sp}$	E_{thr}, Mev	ΔE, Mev
U^{238}	1.30	2.24	5.8	0.6
U^{235}	1.25	2.2	5.75	0.6
Pu^{239}	1.30	2.17	5.48	0.5

It is clear from the table that in U^{238} the fission threshold with the meson present is higher than the excitation energy, while in Pu^{239} it is approximately 0.3 Mev below the excitation energy. Nuclear fission induced by μ mesons via the mechanism discussed here has been studied[6,7] in U^{238}. From this calculation it is clear that Pu^{239} is more suitable for an investigation of this effect.

In conclusion, the authors express their profound gratitude to D. P. Grechukhin for his advice and counsel, and also to V. K. Saul'ev for programming and carrying out the calculation on the electronic computer.

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*ch = cosh; cth = coth; sh = sinh.

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OBSERVATION OF RESONANCE ABSORPTION OF THE 23.8-kev GAMMA RAYS OF Sn^{119} *BY USING THE CONVERSION ELECTRONS*

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THE study of resonance absorption of γ rays by nuclei bound in crystals (the Mössbauer effect) is usually done from the attenuation of the beam of γ rays by filters containing nuclei which can be excited resonantly. Thus the resonance absorption effect is observed on a relatively large background of γ radiation transmitted through the absorber. A considerably larger relative effect can be obtained by observing resonant scattering.[1]

It is also of interest to study the resonance by observing the conversion electrons which are emitted in the de-excitation of the resonance level. Such a method has advantages in those cases where the resonance absorption cross section is much greater than the cross section for the photoeffect, and where the internal conversion coefficient is not too small.

We have used this method to investigate the temperature dependence of the resonance absorption of the 23.8-kev γ rays which occur in the decay of the isomeric state of Sn^{119} at an energy of 89 kev ($T_{1/2} = 250$ days). According to our estimates, the resonance absorption cross section for these γ rays is approximately 30 times greater than that for the photoeffect, while the internal conversion coefficient is 6.3. The recording of the conversion electrons emitted in the deexcitation of the nuclei was done in a double lens β spectrometer whose luminosity with fully open entrance and exit diaphragms was 7%. We used a source which was 0.02 mm thick, with an activity of $30\,\mu C$, obtained by irradiation of metallic tin enriched to 94% Sn^{118} with thermal neutrons. The content of Sn^{119} in the source did not exceed 2.3%.

The absorbers were prepared by depositing a thin layer of tin on an aluminum foil by evaporation in vacuum, where we used tin enriched to 75% Sn^{119}, while ordinary tin was used for control measurements. The absorber thickness was ~ 0.1 mg/cm^2. Between the source and absorber we placed a 6 mm thick plate of beryllium to absorb the β radiation from the source, which contains radioactive impurities, and a palladium filter 0.03 mm thick to reduce the x ray intensity. The source and absorber (together with the filters) were placed in an aluminum box which was tied to a cooling rod. The measurements were made over the temperature range from 83 to 373° K.

A chromel-aluminum thermocouple was used for the temperature measurements, with its junction connected directly to the source box. The high sensitivity of the thermocouple (0.01°) enabled us to maintain the temperature to an accuracy of 1 — 2°. The size of the effect for each temperature was determined from the counting rate of conversion electrons in measurements with the enriched absorber. Since the effective depth from which electrons emerge giving a contribution to the conversion line is very small ($< 0.1\ \mathrm{mg/cm^2}$) the counting rate is proportional to the Sn^{119} content of the absorber. Therefore to determine the background we made measurements with an absorber of unenriched tin. In this case, at a temperature of 373° K the "resonance" electrons constituted less than 1% of the total counting rate.

To find the dependence of the effect on temperature, we made several series of measurements, in each of which we measured the counting rate of conversion electrons for definite values of T. The data from one such measurement are shown in Fig. 1, where we show the conversion lines measured with enriched absorber for temperatures of 273, 173, 133, and 83° K (the curves 2, 3, 4, and 5), and, for comparison, we show the curve measured at 383° K with an unenriched absorber (curve 1). The counting rate at the maximum of curve 5 reached 60 counts/min, while at the maximum of curve 1 it did not exceed 16 counts/min.

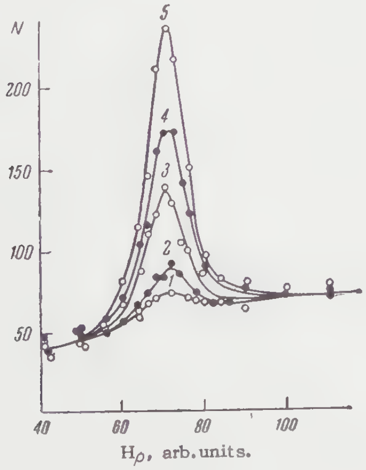

FIG. 1

The temperature dependence of the cross section σ_{res} for resonance absorption is determined by the Debye-Waller factor f. For a source of crystalline material, characterized by a Debye temperature Θ, at temperature T this quantity is given by the expression

$$f = \exp\left\{-\frac{6R}{k\Theta}\left[\frac{1}{4} + \left(\frac{T}{\Theta}\right)^2 \int\limits_0^{\Theta/T} \frac{t\,dt}{e^t - 1}\right]\right\},$$

where R is the recoil energy of the nucleus. If the source and absorber are prepared from the same material, then $\sigma \sim f^2$. Since our measurements did not permit us to determine the absolute value of σ_{res}, because of the large uncertainty in the determination of the effective thickness of the absorber layer which contributes to the conversion line, we tried to determine Θ from the temperature dependence of σ_{res}.

Curves of $f^2(\Theta, T)/f^2(\Theta, T_0)$ were computed for different values of Θ, for T varying over the interval 83 — 383° K. The points T_0 corresponded to temperatures at which measurements were made. After correcting for self-absorption in the source, the relative values of the effect $N_{\mathrm{res}}(T)/N_{\mathrm{res}}(T_0)$ were compared with the theoretical curves for the corresponding value of T_0. The results of the experiment agreed best with the value Θ = 170° K, but there is a systematic deviation of the experimental points from the theoretical curves. This can be seen from Fig. 2, which gives one of the series of curves, corresponding to Θ = 170, 180, and 190° K, for T_0 = 83° K, as well as the measured data. The observed deviation could be explained by a smooth change of Θ with temperature, with Θ ~ 180° at temperatures near that of liquid nitrogen, and Θ ~ 165° for T > 200° K. The computed value f = 0.6 corresponding to Θ

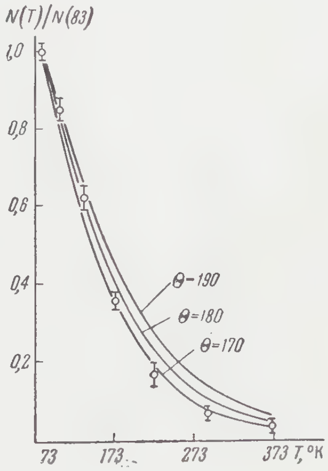

FIG. 2

= 180° is in good agreement with the measurements of other authors.[2,3]

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